

# Latin bitrades, dissections of equilateral triangles and abelian groups

Aleš Drápal\*

Department of Mathematics  
Charles University  
Sokolovská 83  
186 75 Praha 8  
Czech Republic

Carlo Hämmäläinen†

Department of Mathematics  
Charles University  
Sokolovská 83  
186 75 Praha 8  
Czech Republic  
carlo.hamalainen@gmail.com

Vítězslav Kala‡

Department of Mathematics  
Charles University  
Sokolovská 83  
186 75 Praha 8  
Czech Republic

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## Abstract

Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade. With each  $a = (a_1, a_2, a_3) \in T^*$  associate a set of linear equations  $\text{Eq}(T, a)$  of the form  $b_1 + b_2 = b_3$ , where  $b = (b_1, b_2, b_3)$  runs through  $T^* \setminus \{a\}$ . Assume  $a_1 = 0 = a_2$  and  $a_3 = 1$ . Then  $\text{Eq}(T, a)$  has in rational numbers a unique solution  $b_i = \bar{b}_i$ . Suppose that  $\bar{b}_i \neq \bar{c}_i$  for all  $b, c \in T^*$  such that

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$b_i \neq c_i$  and  $i \in \{1, 2, 3\}$ . We prove that then  $T^\Delta$  can be interpreted as a dissection of an equilateral triangle. We also consider group modifications of latin bitrades and show that the methods for generating the dissections can be used for a proof that  $T^*$  can be embedded into the operational table of a finite abelian group, for every spherical latin bitrade  $T$ .

## 1 Introduction

Consider an equilateral triangle  $\Sigma$  that is dissected into a finite number of equilateral triangles. Dissections will be always assumed to be nontrivial, and so the number of dissecting triangles is at least four. Denote by  $a$ ,  $b$  and  $c$  the lines induced by the sides of  $\Sigma$ . It is easy to realize that each side of a dissecting triangle has to be parallel to  $a$  or  $b$  or  $c$ . If  $X$  is a vertex of a dissecting triangle, then  $X$  is a vertex of exactly one, three or six dissecting triangles. Suppose that there is no vertex with six triangles and consider triples  $(u, v, w)$  of lines that are parallel to  $a$ ,  $b$  and  $c$ , respectively, and meet in a vertex of a dissecting triangle that is not a vertex of  $\Sigma$ . The set of all these triples together with the triple  $(a, b, c)$  will be denoted by  $T^*$ , and by  $T^\Delta$  we shall denote the set of all triples  $(u, v, w)$  of lines that are yielded by sides of a dissecting triangle (where  $u$ ,  $v$  and  $w$  are again parallel to  $a$ ,  $b$  and  $c$ , respectively). Observe that the following conditions hold:

- (R1) Sets  $T^*$  and  $T^\Delta$  are disjoint;
- (R2) for all  $(p_1, p_2, p_3) \in T^*$  and all  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ , there exists exactly one  $(q_1, q_2, q_3) \in T^\Delta$  with  $p_r = q_r$  and  $p_s = q_s$ ; and
- (R3) for all  $(q_1, q_2, q_3) \in T^\Delta$  and all  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ , there exists exactly one  $(p_1, p_2, p_3) \in T^*$  with  $q_r = p_r$  and  $q_s = p_s$ .

Note that (R2) would not be true if there had existed six dissecting triangles with a common vertex. Conditions (R1–3) are, in fact, axioms of a combinatorial object called latin bitrades [6, p.148]. This way of their construction was described in [11, 12] and here we shall consider the converse approach, i. e. determining when a latin bitrade yields a dissection. The topic obtained a strong impetus recently when Cavenagh and Lisoněk observed [4] that spherical latin bitrades are equivalent to cubic 3-connected bipartite planar graphs [15]. Since there exists a computer package [2] that uses an algorithm of Batagelj [1] for a fast generation of the latter objects, it is natural to adapt it for generation of dissections. Note that Wanless enumerated small latin bitrades independently of [2]. His results [18] influenced the above mentioned discovery of Cavenagh and Lisoněk.

Some dissections are easy to find, but some are difficult to unravel, and there seems to exist no efficient way to generate dissections directly without resorting to some kind of abstraction. A spherical latin bitrade has to possess an embedding into a cyclic group if it can yield a dissection, but it is not clear if every embedding into a cyclic group can be interpreted as a dissection.

In every dissection of  $\Sigma$  there exist exactly three dissecting triangles with a vertex that is also a vertex of  $\Sigma$ . If we remove one or two of these triangles we obtain a pentagon or a quadrangle. Starting from them we can construct further pentagons or quadrangles by adding a triangle to one of the sides. Any arising trapezoid quadrangle may be completed to a new dissection, and we can regard two dissections as related when they can be built in this manner from a common origin. However, the nature of such origins is not well understood yet. A better understanding might help to explicate the structure of all spherical latin bitrades and to influence a solution of Barnette's conjecture.

Lines parallel to  $a$ ,  $b$  or  $c$  that are induced by a side of a dissecting triangle will be called *dissecting lines*. For any of them the union of all sides of dissecting triangles that are incident to the line forms one or more contiguous segments. If there are two or more segments, then upon the line there exist two dissecting vertices such that all triangles in between are cut by the line into two parts. If such a situation arises for no dissecting line and if no dissecting vertex is incident to six dissecting triangles, then we call the dissection *separated*.

Our initial motivation was to find a relatively efficient algorithm that would produce all separated dissections of small orders. Experiments indicated that for small orders nearly all spherical latin bitrades produce a dissection. The question was how to systematically reverse the process described in the beginning of this section. The first step in the reverse process is the choice of  $u = (u_1, u_2, u_3) \in T^*$  that induces the sides  $a$ ,  $b$  and  $c$  of  $\Sigma$ . With  $u$  chosen we convert  $T = (T^*, T^\Delta)$  into a system of  $s - 1$  linear equations with  $s - 1$  variables, each of which corresponds to a dissecting line. The integer  $s$  coincides with the number of dissecting triangles. The system has always a unique solution and our first main result (Theorem 2.1) states that a *pointed* spherical latin bitrade  $(T, u)$  determines a separated dissection if and only if it is true that inside each group the same value is never assigned to two different unknowns. By a group we mean here the set of all dissecting lines parallel to  $a$  (or  $b$ , or  $c$ ).

The proof of Theorem 2.1 is quite long and stretches over Sections 2 and 3. The result may look to be somewhat counterintuitive since there seems to be no obvious single reason why a solution could not induce a covering of  $\Sigma$  by triangles in which at least one overlapping occurs. We shall use elementary geometry to study the local structure of the overlapping and show that in fact it never occurs.

After proving Theorem 2.1 we realized that some arguments remain valid in a weaker form even when the solution is not necessarily separated. These arguments now appear as the three lemmas of Section 2. They indicate how to prove that every spherical latin bitrade can be embedded into a finite abelian group, our second main result. The additional arguments that are needed for the result appear in Section 4.

The connection of dissections to counting modulo  $n$  becomes clear when we place a dissection upon a grid formed by equilateral triangles of size 1 in such a way that every dissecting triangle becomes a union of basic grid triangles. If the length of the dissected triangle is  $n$ , then the dissecting vertices can be interpreted as cells in the addition table modulo  $n$ . This establishes an

embedding of  $T(*)$  into  $\mathbb{Z}_n(+)$ .

When the solution is not separated, then we do not get an embedding, but we still get a mapping  $T(*) \rightarrow \mathbb{Z}_n(+)$  that behaves like a homomorphism. We shall rather speak about a *homotopy* to stress the fact that the mapping is defined independently for each of the three groups. Lemma 2.4 states that if  $v = (v_1, v_2, v_3) \in T^*$  and  $i \in \{1, 2, 3\}$  are such that  $u_i \neq v_i$ , then there exists a homotopy to a finite cyclic group that assigns to  $u_i$  and  $v_i$  different values unless a certain special situation occurs, and in Section 4 we show that the homotopy can be constructed even in that situation. By taking a product of all such cyclic groups we obtain an embedding of  $T(*)$  into a finite abelian group  $G(+)$ . In Section 5 this observation is put into the context of a general theory describing group modifications of partial quasigroups [8]. We transfer the theory from noncommutative to abelian groups and note that up to isomorphism there exists a unique finite abelian group  $G(+)$  with a natural embedding  $T(*) \rightarrow G(+)$  such that any homotopy  $T(*) \rightarrow H(+)$  can be factorized over this natural embedding. Most of the needed facts are reproved, and that makes the paper practically self-contained.

## 2 Concepts and definitions

We have already mentioned that a dissection of an equilateral triangle is called *separated* if

- (i) no vertex of the dissection is of valence six, and
- (ii) every line  $p$  induced by a side of a dissecting triangle has the property that the subset  $U$  of  $p$  forms a contiguous segment if  $U$  is defined as the union of all sides of dissecting triangles that induce the line  $p$ .

We shall sometimes consider the mates  $T^*$  and  $T^\Delta$  of a latin bitrade  $T$  as two partial quasigroups and write  $a_1 * a_2 = a_3$  when  $(a_1, a_2, a_3) \in T^*$  and  $b_1 \triangle b_2 = b_3$  when  $(b_1, b_2, b_3) \in T^\Delta$ .

A triple of mappings  $(\sigma_1, \sigma_2, \sigma_3)$  will be called a *homotopy* of partial quasigroups  $T(*)$  and  $S(\cdot)$  if  $\sigma_3(u_1 * u_2) = \sigma_1(u_1) \cdot \sigma_2(u_2)$  whenever  $u_1 * u_2$  is defined. Say that  $T(*)$  can be *embedded* into  $S(\cdot)$  if there exists a homotopy  $(\sigma_1, \sigma_2, \sigma_3)$  such that all three mappings  $\sigma_i$  are injective.

Call a latin bitrade  $T = (T^*, T^\Delta)$  *indecomposable* if it cannot be expressed as a disjoint union of two nonempty latin bitrades. We shall investigate only indecomposable latin bitrades.

The number of triples in  $T^*$  is called the *size* of  $T$ . Thus  $s = |T^*| = |T^\Delta|$ . Latin bitrades are often represented by tables as partial latin squares, and so  $a_1$  can be regarded as (a label of) a row,  $a_2$  as a column and  $a_3$  as a symbol, for every  $(a_1, a_2, a_3) \in T^*$ . Denote by  $m$  the aggregate number of rows, columns and symbols. Thus  $m = o_1 + o_2 + o_3$ , where  $o_i = |\{\alpha; \alpha = a_i \text{ for some } (a_1, a_2, a_3) \in T^*\}|$ .

An (indecomposable) latin bitrade is said to be *spherical* if  $m = s + 2$ . This equality can be derived from the Euler identity and expresses the fact that

the bitrade can be represented upon a sphere. In this paper the topological aspects of latin bitrades will be needed only in Section 4 where one can find more information. Nevertheless, we remark here that those indecomposable latin bitrades that yield an oriented surface in a direct way are called *separated*. These are exactly those bitrades in which there does not exist a row (a column, or a symbol) such that one could construct a new bitrade of the same size by dividing all cells of the row (the column, or the symbol) into two new rows (columns, symbols). Spherical bitrades are always separated and the inequality  $m \leq s + 2$  holds for every (indecomposable) latin bitrade.

With each latin bitrade  $T = (T^*, T^\Delta)$  associate a set of equations  $\text{Eq}(T)$  in such a way that every triple  $(a_1, a_2, a_3) \in T^*$  yields the equation  $a_1 + a_2 = a_3$ . The elements  $a_i$  are thus regarded as unknowns. It is natural to have different unknowns for rows, columns and symbols, and so we assume that  $a_i \neq b_j$  whenever  $(a_1, a_2, a_3), (b_1, b_2, b_3) \in T^*$  and  $1 \leq i < j \leq 3$ . (If the condition is violated, then  $T$  can be replaced by an isotopic bitrade for which it is satisfied.)

To build a dissection one needs a latin bitrade  $T = (T^*, T^\Delta)$  and a triple  $a = (a_1, a_2, a_3) \in T^*$ . The pair  $(T, a)$  will be called a *pointed* bitrade. In this section we shall always assume that  $T$  is spherical. We start from the set of equations  $\text{Eq}(T, a)$  that is obtained from  $\text{Eq}(T)$  by removing the equation  $a_1 + a_2 = a_3$ , and by substituting  $a_1 = 0$ ,  $a_2 = 0$  and  $a_3 = 1$ . We get in this way a system with  $m - 3 = s - 1$  variables and  $s - 1$  equations. In Section 5 we shall explain why this system has always a unique solution in rational numbers. If the solution to  $b_i$  is  $u_i$ , where  $(b_1, b_2, b_3) \in T^*$ ,  $b \neq a$ , write  $\tilde{b}_i = u_i$ . In particular,  $\tilde{a}_1 = \tilde{a}_2 = 0$  and  $\tilde{a}_3 = 1$ . Since  $b_1$  is usually associated with a row and  $b_2$  with a column, we associate in the euclidean plane the triple  $b$  with the point  $\bar{b} = (\bar{b}_2, \bar{b}_1)$ . We shall write  $\tilde{b}_1$  to denote the line  $y = \bar{b}_1$ , and further  $\tilde{b}_2$  for the line  $x = \bar{b}_2$  and  $\tilde{b}_3$  for the line  $x + y = \bar{b}_3$ . Note that  $\tilde{a}_1$  is the axial line  $y = 0$ ,  $\tilde{a}_2$  is the axis  $x = 0$  and  $\tilde{a}_3$  is the line  $x + y = 1$ . Denote by  $\Sigma$  the triangle induced by these three lines. Thus  $\Sigma = \{(x, y); x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$ . For each  $c = (c_1, c_2, c_3) \in T^\Delta$  denote by  $\Delta = \Delta(c, a)$  the triangle induced by lines  $\tilde{c}_1$ ,  $\tilde{c}_2$  and  $\tilde{c}_3$ . Of course, it is not clear that  $\Delta$  is really a triangle, i. e. that the lines  $\tilde{c}_i$  do not meet in a single point. If this happens, then we shall say that  $\Delta$  *degenerates*. Our intention is to consider the question of when the triangles  $\Delta(c, a)$ ,  $c \in T^\Delta$ , dissect the triangle  $\Sigma$  and none of them degenerates. In such a dissection each triangle has a side parallel to the axis  $x = 0$  and a side parallel to the axis  $y = 0$ . To obtain a dissection of an equilateral triangle apply the affine transformation  $(x, y) \mapsto (y/2 + x, \sqrt{3}y/2)$ .

Say that the solution to  $\text{Eq}(T, a)$  is *separated* if no two unknowns representing two rows (or two columns, or two symbols) attain the same value. In other words we require that  $\tilde{b}_i = \tilde{d}_i$  if and only if  $b_i = d_i$ , for every  $(b_1, b_2, b_3), (d_1, d_2, d_3) \in T^*$  and every  $i \in \{1, 2, 3\}$ . It is clear that if the solution to  $\text{Eq}(T, a)$  is separated, then  $\Delta(c, a)$  degenerates for no  $c \in T^\Delta$ .

Let us return to the procedure described in the beginning of Section 1, assuming that there is no dissecting vertex of valence six. The dissection of  $\Sigma$  can be then interpreted as a pointed bitrade  $(T, a)$ , where  $a$  is the triple of lines that induce the sides of  $\Sigma$ . Denote by  $s$  the number of dissecting triangles and

by  $\ell$  the number of contiguous segments. We shall also use  $m$  to denote the aggregate number of rows, columns and symbols. We see that  $s$  is equal to the size of  $T$ , that  $m \leq \ell$ , and that the equality  $m = \ell$  holds if and only if the dissection is separated. Furthermore, there are  $s + 2$  dissecting vertices, and each of them is an extreme point of exactly two edges. Thus  $\ell = s + 2$ , and  $T$  is spherical if and only if the dissection is separated.

We can assume that  $\Sigma = \{(x, y); x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$ . The dissection yields a solution to  $\text{Eq}(T, a)$ . Since there is only one solution, we see that the dissection can be reconstructed as  $\{\Delta(c, a); c \in T^\Delta\}$ . A separated dissection thus induces a separated solution to  $\text{Eq}(T, a)$ . Theorem 2.1 claims that this observation can be reversed. We shall prove it in Section 3. (Nearly all proofs in Sections 2–4 depend upon the fact that the linear system  $\text{Eq}(T, a)$  always has a unique solution. That was proved already in [10] and we prove it anew in Section 5, in Lemma 5.3.)

**Theorem 2.1** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade, and suppose that  $a = (a_1, a_2, a_3) \in T^*$  is such a triple that the solution to  $\text{Eq}(T, a)$  is separated. Then the set of all triangles  $\Delta(c, a)$ ,  $c \in T^\Delta$ , dissects the triangle  $\Sigma = \{(x, y); x \geq 0, y \geq 0 \text{ and } x + y \leq 1\}$ . This dissection is separated.*

The construction of latin bitrades from triangle dissections was first described in [11]. It is also discussed in [16], and [12] gives a topological interpretation. By [12] one can derive a spherical latin bitrade from dissections with values of valence six as well. That opens a possibility to extend Theorem 2.1 to all dissections.

For an example of a dissection, consider the following spherical bitrade  $(T^*, T^\Delta)$ :

$T^* =$	*	$c_0$	$c_1$	$c_2$	$c_3$	$c_4$
	$r_0$	$s_4$		$s_0$		$s_2$
	$r_1$				$s_2$	$s_4$
	$r_2$	$s_0$	$s_1$	$s_2$	$s_3$	
	$r_3$	$s_1$	$s_3$		$s_4$	

$T^\Delta =$	$\Delta$	$c_0$	$c_1$	$c_2$	$c_3$	$c_4$
	$r_0$	$s_0$		$s_2$		$s_4$
	$r_1$				$s_4$	$s_2$
	$r_2$	$s_1$	$s_3$	$s_0$	$s_2$	
	$r_3$	$s_4$	$s_1$		$s_3$	

Let  $a = (a_1, a_2, a_3) = (r_0, c_0, s_4)$ . Then the system of equations  $\text{Eq}(T, a)$  has the solution

$$\begin{aligned}\bar{r}_0 &= 0, \bar{r}_1 = 2/7, \bar{r}_2 = 5/14, \bar{r}_3 = 4/7 \\ \bar{c}_0 &= 0, \bar{c}_1 = 3/14, \bar{c}_2 = 5/14, \bar{c}_3 = 3/7, \bar{c}_4 = 5/7 \\ \bar{s}_0 &= 5/14, \bar{s}_1 = 4/7, \bar{s}_2 = 5/7, \bar{s}_3 = 11/14, \bar{s}_4 = 1.\end{aligned}$$

The dissection is shown in Figure 1. Entries of  $T^\Delta$  correspond to triangles in the dissection. For example,  $(r_0, c_0, s_0) \in T^\Delta$  is the triangle bounded by the lines  $y = \bar{r}_0 = 0$ ,  $x = \bar{c}_0 = 0$ ,  $x + y = \bar{s}_0 = 5/14$  while  $(r_1, c_3, s_2) \in T^*$  corresponds to the intersection of the lines  $y = \bar{r}_1 = 2/7$ ,  $x = \bar{c}_3 = 3/7$ ,  $x + y = \bar{s}_2 = 5/7$ .

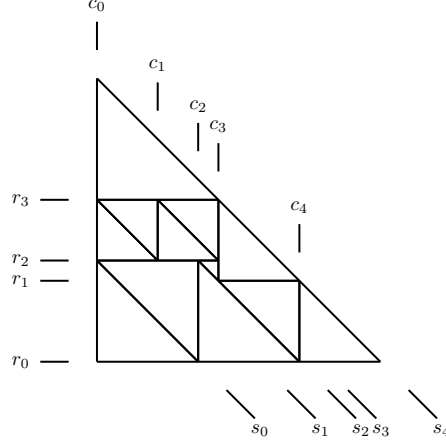


Figure 1: Dissection for a spherical bitrade. The labels  $r_i$ ,  $c_j$ ,  $s_k$ , refer to lines  $y = \bar{r}_i$ ,  $x = \bar{c}_j$ ,  $x + y = \bar{s}_k$ , respectively.

We shall conclude this section by three preparatory lemmas. All of them assume that  $T = (T^*, T^\Delta)$  is a spherical latin bitrade and that  $a = (a_1, a_2, a_3)$  is a fixed element of  $T^*$ .

By a *lateral* point of a triangle we understand a point that is upon a side of the triangle, but not equal to a vertex. Each triangle is thus a disjoint union of its vertices, lateral points and interior points.

In Lemmas 2.2 and 2.4 we shall use permutations  $\mu_{r,s}$  of  $T^\Delta$ , for every  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ . Each of the mappings  $\mu_{r,s}$  permutes the set  $T^\Delta$  in such a way that for every  $c = (c_1, c_2, c_3) \in T^\Delta$  we take first  $d = (d_1, d_2, d_3) \in T^*$  such that  $d_t = c_t$  for  $t \neq s$ , and then we choose  $c' = \mu_{r,s}(c) \in T^\Delta$  so that  $c' = (c'_1, c'_2, c'_3)$  satisfies  $c'_t = d_t$  for  $t \neq r$ . Note that by reversing the process we get  $c = \mu_{s,r}(c')$ . Hence  $\mu_{s,r} = \mu_{r,s}^{-1}$ . Note that if  $t \in \{1, 2, 3\}$  is different from both  $r$  and  $s$ , then  $c'_t = c_t$  and  $\mu_{s,r}(c) = \mu_{s,t}\mu_{t,r}(c)$ .

**Lemma 2.2** *Assume that  $b = (b_1, b_2, b_3) \in T^*$ , and that  $1 \leq i < j \leq 3$ . If  $\bar{b}$  is not a vertex of  $\Sigma$ , then there exists  $c = (c_1, c_2, c_3) \in T^\Delta$  such that  $\Delta(c, a)$  does not degenerate,  $\bar{b}_i = \tilde{c}_i$  and  $\bar{b}_j = \tilde{c}_j$ .*

*Proof.* We shall say that  $c \in T^\Delta$  *degenerates at  $b$*  if  $\Delta(c, a)$  degenerates and the lines  $\tilde{c}_1$ ,  $\tilde{c}_2$  and  $\tilde{c}_3$  meet in the point  $\bar{b}$ . Denote by  $C$  the set of all such  $c \in T^\Delta$ . If  $C = \emptyset$ , then the lemma is obvious. Let us have  $C$  nonempty, and suppose first that there exist  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ , such that  $\mu_{r,s}(c) \notin C$  for some  $c \in C$ .

While investigating this case we shall assume that the triples  $d$ ,  $c$  and  $c'$  have the same meaning as in the definition of  $\mu_{r,s}$  and that  $\{1, 2, 3\} = \{r, s, t\}$ . Since  $c$  degenerates at  $b$ , there must be  $\bar{b} = \bar{d}$ . We assume that  $\bar{b}$  is not a vertex of  $\Sigma$ , and hence  $d \neq a$ . Thus  $\tilde{b}_t = \tilde{d}_t = \tilde{c}'_t$  and  $\tilde{b}_s = \tilde{d}_s = \tilde{c}'_s$ , by the construction of  $c'$ . Since  $\Delta(c', a)$  does not degenerate, we are done when  $\{i, j\} = \{s, t\}$ . By

switching  $r$  and  $s$  we also obtain immediately the case  $\{i, j\} = \{r, t\}$  since from  $\mu_{s,r} = \mu_{r,s}^{-1}$  we see that there exists  $e \in C$  such that  $\mu_{s,r}(e) \notin C$ . This means that we also have a proof for the case  $\{i, j\} = \{r, s\}$  when there exists  $e \in C$  such that  $\mu_{t,s}(e) \notin C$  or  $\mu_{r,t}(e) \notin C$ . However, we can always put  $e = c$  or  $e = \mu_{t,s}(c)$  since  $\mu_{r,s}(c) = \mu_{r,t}\mu_{t,s}(c) \notin C$ .

Suppose now that  $C$  is nonempty and that  $\mu_{r,s}(c) \in C$  for every  $c \in C$  and  $r, s \in \{1, 2, 3\}$ ,  $r \neq s$ . Define  $D$  as the set of all  $d = (d_1, d_2, d_3) \in T^*$  such that  $d$  agrees in two coordinates with some  $c \in C$ . Choose  $d \in D$  and  $r \in \{1, 2, 3\}$ . Let  $c' \in T^\Delta$  agree with  $d$  in the two coordinates different from the  $r$ th coordinate. We shall prove that  $c' \in C$ . There exists some  $c \in C$  that agrees with  $d$  in two coordinates. If  $c' = c$ , then there is nothing to prove. We can thus assume that  $c_t = c'_t = d_t$ ,  $c'_s = d_s \neq c_s$  and  $c_r = d_r \neq c'_r$ , where  $\{r, s, t\} = \{1, 2, 3\}$ . But then  $c' = \mu_{r,s}(c)$  and we have  $c' \in C$  by our assumption. We see that  $(D, C)$  is a latin bitrade. Every  $c \in C$  degenerates at  $b$ . Since  $\bar{b}$  is not a vertex of  $\Sigma$ , there cannot be  $a \in D$ . Hence  $(C, D)$  does not coincide with  $T = (T^*, T^\Delta)$ . This is a contradiction because  $T$  is assumed to be indecomposable.  $\square$

**Lemma 2.3** *The triangle  $\Delta(c, a)$  is contained in the triangle  $\Sigma$ , for every  $c = (c_1, c_2, c_3) \in T^\Delta$ .*

*Proof.* Put  $H_1 = \{(x, y); y < 0\}$ ,  $H_2 = \{(x, y); x < 0\}$  and  $H_3 = \{(x, y); x + y > 1\}$ . Our task is to obtain a contradiction when  $\bar{b} \in H_i$  for some  $b = (b_1, b_2, b_3) \in T^*$ , where  $b \neq a$  and  $i \in \{1, 2, 3\}$ . The three cases are similar and so we can assume that there exists  $b$  with  $\bar{b} \in H_3$ . Put  $h = \max\{\bar{b}_3; \bar{b} \in H_3\}$  and choose  $b \in T^*$  such that  $\bar{b}_3 = h$  and  $\bar{b}_1$  attains the maximum possible value. By Lemma 2.2 there exists  $c = (c_1, c_2, c_3) \in T^\Delta$  such that  $\Delta = \Delta(c, a)$  does not degenerate,  $\bar{c}_3 = h = \bar{b}_3$  and  $\bar{c}_1 = \bar{b}_1$ . Let  $d, e \in T^*$  be such that  $d = (d_1, c_2, c_3)$  and  $e = (c_1, c_2, e_3)$ , for some  $d_1$  and  $e_3$ . Then  $\bar{b}$ ,  $\bar{d}$  and  $\bar{e}$  are the (pairwise distinct) vertices of  $\Delta$ . The vertex  $\bar{d}$  is upon the line  $\bar{c}_3 = \bar{b}_3$ , and hence  $h = \bar{d}_1 + \bar{c}_2 < \bar{b}_1 + \bar{c}_2$ , by the maximality of  $\bar{b}_1$ . The vertex  $\bar{e}$  is upon the line  $\bar{b}_1 = \bar{c}_1$ , and hence  $h > \bar{c}_1 + \bar{c}_2 = \bar{b}_1 + \bar{c}_2$ . We have obtained a contradiction.  $\square$

**Lemma 2.4** *Suppose that  $b = (b_1, b_2, b_3) \in T^*$  and  $i \in \{1, 2, 3\}$  are such that  $b_i \neq a_i$ . If  $\bar{b}_i = \bar{a}_i$ , then  $\Delta(c, a)$  degenerates whenever  $c = (c_1, c_2, c_3) \in T^\Delta$  and  $c_i = b_i$ . The point  $\bar{b}$  is never a vertex of  $\Sigma$ .*

*Proof.* Let us consider the case  $i = 1$ ; the other cases are similar. Thus  $b_1 \neq a_1$  and  $\bar{b}_1 = \bar{a}_1 = 0$ . If  $b' = (b_1, b'_2, b'_3) \in T^*$ , then  $\bar{b}'_3 = \bar{b}_1 + \bar{b}'_2 = \bar{b}'_2$  since  $b' \neq a$ . Put  $H = \{\bar{b}'_2; (b_1, b'_2, b'_3) \in T^*\}$  and choose  $h \in H$ .

Then there certainly exists at least one  $c = (b_1, c_2, c_3) \in T^\Delta$  such that  $\bar{c}_2 = h$ . Every  $c' = (b_1, c'_2, c'_3) \in T^\Delta$  can be expressed as  $\mu_{3,2}^i(c)$  for some  $i \geq 0$  since the bitrade  $T$  is separated. Can  $c$  be chosen in such a way that  $\Delta(c, a)$  does not degenerate? If  $\Delta(c, a)$  degenerates and  $c' = \mu_{3,2}(c)$ , then  $\bar{c}'_2 = \bar{c}_3 = \bar{c}_2 = h$  as well since  $(b_1, c'_2, c_3) \in T^*$ . Therefore either  $H = \{h\}$ , or the sought choice is possible.



Let us thus suppose that  $\Delta(c, a)$  does not degenerate. Then  $\bar{c}_2 \neq \bar{c}_3$  and  $H$  contains at least two elements. Assume that  $h = \max H$  and that  $c'$  equals  $\mu_{3,2}(c)$ . Both  $(\bar{c}_2, 0) = (h, 0)$  and  $(\bar{c}'_2, 0)$  are vertices of  $\Delta(c, a)$ . The third vertex is equal to  $(\bar{c}_2, \bar{d}_1)$ , where  $d = (d_1, c_2, c_3) \in T^*$ . There cannot be  $d = a$  since  $\bar{c}_2 = h > 0$ . From Lemma 2.3 we get  $\bar{d}_1 > 0$ , and hence  $h = \bar{c}'_2 = \bar{c}_3 = \bar{d}_1 + \bar{c}_2 > \bar{c}_2 = h$ , which contradicts the choice of  $h$ . We have proved that  $H$  always contains only one element. That means that  $\Delta(c, a)$  always degenerates when  $c = (c_1, c_2, c_3) \in T^\Delta$ ,  $c_1 \neq a_1$  and  $\bar{c}_1 = 0$  (more generally, when  $c_i \neq a_i$  and  $\bar{c}_i = \bar{a}_i$ ).

What remains is to exclude cases  $H = \{0\}$  and  $H = \{1\}$ . Each of them implies that the set  $K_j = \{(u_1, u_2, u_3) \in T^*; |\bar{u}_j - \bar{a}_j| = 1\}$  is nonempty for some  $j \in \{1, 2, 3\}$ . We shall show that this never occurs.

We shall first do so under an additional assumption that the set  $C = \{c \in T^\Delta; \Delta(c, a) \text{ does not degenerate and } \Delta(c, a) \neq \Sigma\}$  is nonempty. Put  $\Gamma = \bigcup(\Delta(c, a); c \in C)$  and set  $h_1 = \min\{y; (x, y) \in \Gamma\}$  and  $h_2 = \min\{x; (x, h_1) \in \Gamma\}$ . The point  $(h_2, h_1)$  must be a vertex of some  $\Delta(e, a)$ ,  $e \in C$ . Hence there exists  $b = (b_1, b_2, b_3) \in T^*$  with  $\bar{b}_1 = h_1$ ,  $\bar{b}_2 = h_2$ , and  $\bar{b}_3 = h_1 + h_2$ . Assume  $(h_2, h_1) \neq (0, 0)$ . From Lemma 2.3 we see that  $(h_2, h_1)$  is not a vertex of  $\Sigma$ , and from Lemma 2.2 we obtain the existence of  $c = (c_1, c_2, c_3) \in C$  with  $\bar{c}_1 = h_1$  and  $\bar{c}_3 = h_1 + h_2$ . Any triangle with sides upon  $\bar{c}_1$  and  $\bar{c}_3$  contains an element  $(x, y)$  such that either  $y < h_1$ , or  $y = h_1$  and  $x < h_2$ . Hence  $(h_1, h_2) = (0, 0)$  and there exists  $e = (e_1, e_2, e_3) \in C$  with  $\bar{e}_1 = \bar{e}_2 = 0$ . By the first part of the proof,  $\Delta(e, a)$  degenerates if  $e_1 \neq a_1$  or  $e_2 \neq a_2$ . However,  $\Delta(e, a)$  does not degenerate as  $e \in C$ . Hence  $e = (a_1, a_2, e_3)$ .

We shall now use the fact that  $\Delta(e, a)$  does not degenerate to prove that  $K_3 = \emptyset$ . Assume  $K_3 \neq \emptyset$  and note that  $K_3 = \{(u_1, u_2, u_3) \in T^*; \bar{u}_3 = 0\}$ . To get a contradiction we shall use the indecomposability of  $T$ . To prove that  $K_3$  induces a decomposition of  $T$  into two latin bitrades it suffices to verify that  $\bar{u}'_3 = 0$  whenever  $u = (u_1, u_2, u_3) \in K_3$ ,  $v = (v_1, v_2, v_3) \in T^\Delta$  and  $u' = (u'_1, u'_2, u'_3) \in T^*$  are such that both pairs  $\{u, v\}$  and  $\{v, u'\}$  disagree in exactly one coordinate. There is nothing to prove if  $u'_3 = u_3$ . The first two coordinates can be treated similarly, and so we can assume that  $u_1 = u'_1 = v_1$ . Thus  $u' = \mu_{3,2}(u)$  or  $u' = \mu_{2,3}(u)$ . It suffices to consider only the former case since  $\mu_{2,3}(u)$  can be obtained by iterative applications of  $\mu_{3,2}$  which start at  $u$  as  $\mu_{2,3} = \mu_{3,2}^{-1}$ .

Let us have  $u' = \mu_{3,2}(u)$ . Then  $u_3 = v_3$ ,  $u_1 = v_1 = u'_1$  and  $v_2 = u'_2$ . Two vertices of  $\Delta(v, a)$  are upon the line  $\bar{v}_3 = \bar{u}_3$ . The points of the line satisfy  $x + y = 0$ . Both vertices are thus equal to  $(0, 0)$ , by Lemma 2.3, and hence  $\bar{u}'$ , the third vertex of  $\Delta(v, a)$ , is equal to  $(0, 0)$  as well. Therefore either  $\bar{u}'_3 = 0$ , or  $u' = a$ . If  $u' = a$ , then  $v_1 = a_1$  and  $v_2 = a_2$ . Thus  $v$  coincides with the triple  $e$  of the preceding paragraph. However, we cannot have  $e = v$  since  $\Delta(v, a)$  degenerates while  $\Delta(e, a)$  does not. Hence  $\bar{u}'_3 = 0$  in all cases.

Finally, let us suppose that the set  $C$  is empty. If  $\Delta(c, a) = \Sigma$  for some  $c = (c_1, c_2, c_3) \in T^\Delta$ , then  $\bar{c}_i = \bar{a}_i$  for all  $i \in \{1, 2, 3\}$ . In at least one case there must be  $c_i \neq a_i$  since  $c \neq a$ . Hence  $\Delta(c, a)$  has to degenerate, by the first part of the proof. That is a contradiction, and therefore what remains is to solve

the case when  $\Delta(c, a)$  degenerates for every  $c \in (c_1, c_2, c_3) \in T^\Delta$ . Every such  $c$  thus satisfies  $\bar{c}_1 + \bar{c}_2 = \bar{c}_3$ . If  $c = (a_1, c_2, c_3) \in T^\Delta$ ,  $d = (a_1, c_2, d_3) \in T^*$  and  $c' = (a_1, d_2, d_3) \in T^\Delta$ , then  $c_2 \neq a_2$  implies  $\bar{d}_2 = \bar{d}_3 = \bar{c}_2 = \bar{c}_3$  since  $\bar{a}_1 = 0$ . The arrows  $c \rightarrow c' = \mu_{2,3}(c)$  thus retain the value  $\bar{c}_2 = \bar{c}_3$  for all  $(a_1, c_2, c_3) \in T^\Delta$ , with a possible exception of one such  $c$ . However, the arrows form a cycle and so there is no exception. By setting  $d = a$  we see that there exists a cycle of  $\mu_{2,3}$  in which both  $c_2 = a_2$  and  $c_3 = a_3$  occur. That supplies a contradiction since  $\bar{a}_2 \neq \bar{a}_3$ . Hence there always exists  $c \in T^\Delta$  that does not degenerate.  $\square$

### 3 Separated solutions yield dissections

Suppose that the conditions of Theorem 2.1 are satisfied. The triple  $a = (a_1, a_2, a_3) \in T^*$  is fixed throughout the section and so we shall write  $\Delta(c, a)$  simply as  $\Delta(c)$ , for every  $c \in T^\Delta$ . Our goal is to show that these triangles dissect the triangle  $\Sigma$ . Denote the vertices of  $\Sigma$  by  $A = (0, 0)$ ,  $B = (0, 1)$  and  $C = (1, 0)$ . By Lemma 2.3,  $\Delta \subseteq \Sigma$  for all  $\Delta(c)$ ,  $c \in T^\Delta$ .

In the rest of this section we shall call elements of  $\{\Delta(c); c \in T^\Delta\}$  just *triangles*. None of them degenerates because the solution to  $\text{Eq}(T, a)$  is assumed to be separated. The union of all triangle sides will be denoted by  $\Pi$ , and we shall be investigating the set  $\Gamma = (\text{Int } \Sigma) - \Pi$ . For each  $X \in \Gamma$  denote by  $\pi(X)$  the number of triangles in which  $X$  is an interior point. The set  $\Gamma$  is open and we clearly have  $\pi(X) = \pi(Y)$  when  $X$  and  $Y$  are in the same component of  $\Gamma$ . We wish to show that the triangles dissect  $\Sigma$ , and that is the same as proving  $\pi(X) = 1$  for each  $X \in \Gamma$ . We shall assume that

$$\hat{\Gamma} = \{X \in \Gamma; \pi(X) \neq 1\} \neq \emptyset,$$

and argue by contradiction. The set  $\hat{\Gamma}$  is open as well, and its closure is compact. Therefore there exists a unique point  $P = (\alpha, \beta)$  in the closure of  $\hat{\Gamma}$  such that if  $P' = (\alpha', \beta')$  is another point of the closure, then either (1)  $\alpha + \beta < \alpha' + \beta'$ , or (2)  $\alpha + \beta = \alpha' + \beta'$  and  $\beta < \beta'$ .

If  $P$  is a lateral point of  $\Sigma$ , then  $P$  is clearly incident to one of the two axes. Furthermore, in such a case  $P$  has to be a vertex of a triangle since otherwise we might move  $P$  towards  $A$ .

For every  $X \in \Gamma$  there exists a neighbourhood  $U \subseteq \Gamma$  in which  $\pi$  is a constant function. Hence  $P \notin \Gamma$ . It follows that  $P \in \Pi$  since either  $P \in (\text{Int } \Sigma) \setminus \Gamma$ , or  $P \in \Sigma \setminus (\text{Int } \Sigma)$ .

**Lemma 3.1** *The point  $P$  does not equal  $A$ ,  $B$  or  $C$ .*

*Proof.* Clearly  $P \notin \{B, C\}$ , by the definition of  $P$ . Assume  $P = A$  and consider the triple  $a' = (a_1, a_2, a'_3) \in T^\Delta$  (recall that  $\tilde{a}_1$  is the horizontal axis and  $\tilde{a}_2$  is the vertical axis). All points  $X \in \Gamma$  that are close enough to  $A$  belong to  $\Delta(a')$  and that means that  $\pi(X) \geq 1$ . We assume  $A = P$ , and so there must exist another triangle for which  $A$  is a vertex. However, that is impossible since such

a triangle would have to be of the form  $\Delta(a'')$ , where  $a'' = (a_1, a_2, a_3'') \in T^\Delta$  and  $a_3' \neq a_3''$ .  $\square$

Denote by  $p_1, p_2$  and  $p_3$  the lines that pass through  $P$  and are parallel to  $\tilde{a}_1, \tilde{a}_2$  and  $\tilde{a}_3$ , respectively. By Lemma 3.1 the existence of a triple  $b = (b_1, b_2, b_3) \in T^*$  such that  $\tilde{b}_i = p_i$  for all  $i \in \{1, 2, 3\}$  is equivalent to the fact that  $P$  is a triangle vertex.

The next lemma is an immediate consequence of the assumption that the solution to  $\text{Eq}(T, a)$  is separated.

**Lemma 3.2** *Assume that  $P$  is a triangle vertex. Then for each  $i$  and  $j$ , where  $1 \leq i < j \leq 3$ , there exists a unique  $c = (c_1, c_2, c_3) \in T^\Delta$  such that  $\tilde{c}_i = p_i$  and  $\tilde{c}_j = p_j$ . We shall denote  $\Delta(c)$  by  $\Delta(i, j)$ .*

**Lemma 3.3** *The point  $P$  is not a lateral point of  $\Sigma$ .*

*Proof.* Assume the contrary. Since  $P$  cannot be upon  $\tilde{a}_3$ , it has to be an axial point, as we have already remarked in the text above. Denote by  $p$  the axis that passes through  $P$  and by  $p^+$  the half-plane induced by  $p$  that contains  $\Sigma$ . If  $P$  were not a triangle vertex, we could move  $P$  along  $p$  towards  $A$ . Hence  $P$  is a triangle vertex. By Lemma 3.2 for all  $i$  and  $j$  with  $1 \leq i < j \leq 3$  there exists a triangle  $\Delta(i, j)$ . Clearly  $\Delta(i, j) \subset p^+$ . We can thus find a neighbourhood  $U$  of  $P$  such that  $U \cap \Sigma \subset \Delta(1, 2) \cup \Delta(1, 3) \cup \Delta(2, 3)$ . No two of these three triangles share an interior point and no other triangle has  $P$  as a vertex. Since  $P$  is in the closure of  $\hat{\Gamma}$ , there must exist a triangle  $\Delta$  that intersects every neighbourhood of  $P$ . We see that  $P$  is a lateral point of  $\Delta$ , and therefore there exists a neighbourhood  $U$  of  $P$  such that  $U \cap \Delta \cap \Delta(i, j) \neq \emptyset$  for all choices of  $i$  and  $j$ . That makes possible a move of  $P$  within  $U$  along  $p$  towards  $A$ , and that is our concluding contradiction.  $\square$

We can thus assume that  $P$  is an interior point of  $\Sigma$ . Choose a circular neighbourhood of  $P$  and consider the six sectors of the circle that are induced by lines  $p_i$ ,  $1 \leq i \leq 3$ . Call the sectors alternately *positive* and *negative* so that we regard as positive the sector formed by all points  $(\alpha', \beta')$  with  $\alpha' \geq \alpha$  and  $\beta' \geq \beta$ . Of the two sectors bordered by  $p_i$  and  $p_{i+1}$  denote the positive by  $C_{i,i+1}$  and the negative by  $C_{i+1,i}$ . The indices are computed modulo 3 and the neighbourhood is chosen small enough to have  $\text{Int } C_{ij} \subseteq \Gamma$ . Thus  $\pi(X) = \pi(X')$  when there exist  $i, j \in \{1, 2, 3\}$  such that both  $X$  and  $X'$  belong to  $\text{Int } C_{ij}$ , and we shall denote this integer by  $\pi_{ij}$ . The arrangement of sectors is illustrated by Figure 2.

For  $i \in \{1, 2, 3\}$  denote by  $p_i^+$  the half-plane determined by  $p_i$  that contains  $C_{i,i+1} \cup C_{i-1,i+1} \cup C_{i-1,i}$ , and by  $p_i^-$  the opposite half-plane.

**Lemma 3.4** *If  $(i, j) \in \{(3, 1), (2, 1), (2, 3), (1, 3)\}$ , then  $\pi_{ij} = 1$ .*

*Proof.* Suppose first that  $\pi_{ij} \geq 2$  or  $\pi_{ij} = 0$ , where  $(i, j) \in \{(3, 1), (2, 1), (2, 3)\}$ . Then  $\text{Int } C_{ij} \subseteq \hat{\Gamma}$  and  $\alpha' + \beta' < \alpha + \beta$  for every  $(\alpha', \beta') \in \text{Int } C_{ij}$ . If  $\pi_{13} \geq 2$  or

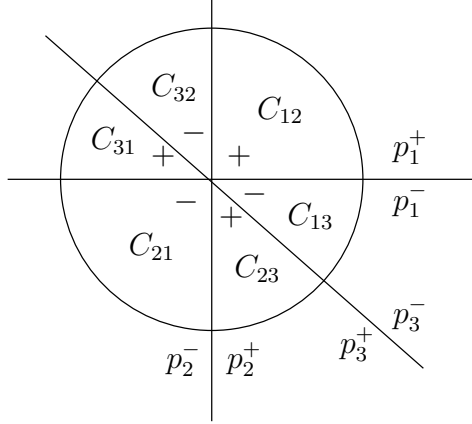


Figure 2: The sectors and half-planes induced by the point  $P$ .

$\pi_{13} = 0$ , then  $\alpha' + \beta' = \alpha + \beta$  and  $\beta' < \beta$  for every point  $(\alpha', \beta') \in C_{13} \cap p_3$  that differs from  $P$ . In every case we thus obtain a contradiction to the choice of  $P$ .  $\square$

There may exist a triangle  $\Delta$  such that  $C_{ij} \subset \Delta$  and  $P$  is a vertex of  $\Delta$ . If such a triangle exists, then it is determined uniquely, by Lemma 3.2, and we shall denote it by  $\Delta_{ij}$ . If the triangle exists, then it is equal to  $\Delta(i, j)$  when  $i < j$ , and to  $\Delta(j, i)$  when  $j < i$ . Note that if  $\Delta_{ij}$  exists, then  $\Delta_{ji}$  does not exist.

By writing  $\Delta \subset p_i^\pm$  we mean that either  $\Delta \subset p_i^+$ , or  $\Delta \subset p_i^-$ . If the point  $P$  is a lateral point of a triangle  $\Delta \subset p_i^\pm$ , then  $C_{rs} \subset \Delta$  for every  $C_{rs} \subset p_i^\pm$ . Furthermore,  $\pi_{rs} = 1$  for at least one such  $(r, s)$ , by Lemma 3.4. This means that  $\Delta$ , if it exists, is uniquely determined, and we shall denote it by  $\Delta_i^\pm$  (the existence of  $\Delta_i^+$  need not exclude the existence of  $\Delta_i^-$ ).

For formal reasons it is sometimes useful to write  $p_i^\varepsilon$  in place of  $p_i^+$  or  $p_i^-$ , where  $\varepsilon \in \{-1, 1\}$ . Under this notation

$$p_i^\varepsilon \supset C_{i,i+\varepsilon} \cup C_{i-\varepsilon,i+\varepsilon} \cup C_{i-\varepsilon,i} \text{ for every } \varepsilon \in \{-1, 1\} \text{ and } i \in \{1, 2, 3\}.$$

**Lemma 3.5** *Let  $i \in \{1, 2, 3\}$  and  $\varepsilon \in \{-1, 1\}$  be such that both  $\Delta_{i,i+\varepsilon}$  and  $\Delta_{i-\varepsilon,i}$  exist and neither of them is equal to  $\Delta_{32}$  or  $\Delta_{12}$ . If  $\pi_{i-\varepsilon,i+\varepsilon} \geq 1$ , then the triangle  $\Delta_{i-\varepsilon,i+\varepsilon}$  exists as well.*

*Proof.* Let  $\Delta$  be a triangle that contains  $C_{i-\varepsilon,i+\varepsilon}$ , and assume that  $\Delta_{i-\varepsilon,i+\varepsilon}$  does not exist. Then  $P$  has to be a lateral or interior point of  $\Delta$ , and so  $\Delta$  contains  $C_{i,i+\varepsilon}$  or  $C_{i-\varepsilon,i}$ . Hence  $\pi_{i,i+\varepsilon} \geq 2$  or  $\pi_{i-\varepsilon,i} \geq 2$ , and that contradicts our assumptions, by Lemma 3.4.  $\square$

**Lemma 3.6** *Suppose that  $P$  is a vertex of a triangle. Then for some  $i \in \{1, 2, 3\}$  and  $\varepsilon \in \{-1, 1\}$  there exist triangles  $\Delta_{i,i+\varepsilon}$ ,  $\Delta_{i-\varepsilon,i+\varepsilon}$ ,  $\Delta_{i-\varepsilon,i}$  and  $\Delta_i^{-\varepsilon}$ .*

*Proof.* By Lemma 3.2 there exist exactly three triangles of the form  $\Delta_{jk}$ . Suppose that they are not contained in any half-plane  $p_i^\varepsilon$ . Then they have to correspond either to all sectors  $C_{j,j+1}$  (the positive sectors), or to all sectors  $C_{j,j-1}$  (the negative sectors). However, in both these cases we easily obtain a contradiction by means of Lemmas 3.5 and 3.4. Therefore there exists a unique pair  $(i, \varepsilon)$  such that all three triangles with vertex  $P$  are contained in  $p_i^\varepsilon$ . Note that the sectors  $C_{i,i+\varepsilon} \cup C_{i,i-\varepsilon}$  and  $C_{i+\varepsilon,i} \cup C_{i-\varepsilon,i}$  are opposite, and that they are separated by interspersed sectors  $C_{i-\varepsilon,i+\varepsilon}$  and  $C_{i+\varepsilon,i-\varepsilon}$ . Hence no union of two adjacent sectors  $C_{rs}$  intersects both  $C_{i,i+\varepsilon} \cup C_{i,i-\varepsilon}$  and  $C_{i+\varepsilon,i} \cup C_{i-\varepsilon,i}$ . It follows that one of them has no interior point common with the sector  $C_{12} \cup C_{32}$ . Thus  $\pi_{i,i+\varepsilon} = \pi_{i,i-\varepsilon} = 1$  or  $\pi_{i+\varepsilon,i} = \pi_{i-\varepsilon,i} = 1$ , by Lemma 3.4. Let  $\Delta$  be the unique triangle that contains  $C_{i,i-\varepsilon}$  in the former case, and  $C_{i+\varepsilon,i}$  in the latter case. The triangle  $\Delta$  has been chosen in such a way that it is not contained in  $p_i^\varepsilon$ . All three triangles for which  $P$  is a vertex are contained in  $p_i^\varepsilon$ . Hence  $P$  is not a vertex of  $\Delta$  and has to be a lateral or interior point of  $\Delta$ . We shall choose a triangle  $\Delta' \subset p_i^\varepsilon$  for which  $P$  is a vertex and which overlaps with no other triangle upon a neighbourhood of  $P$ . In the former case put  $\Delta' = \Delta_{i,i+\varepsilon}$  (and use  $\pi_{i,i+\varepsilon} = 1$ ), while in the latter case set  $\Delta' = \Delta_{i-\varepsilon,i}$  (and use  $\pi_{i-\varepsilon,i} = 1$ ). Triangles  $\Delta$  and  $\Delta'$  have been defined in such a way that  $\Delta \cap \Delta' \cap p_i$  is a nontrivial segment. The choice of  $\Delta'$  guarantees that they do not overlap, and therefore  $p_i$  has to induce a side of  $\Delta$ . This means that  $P$  is a lateral point of  $\Delta$ , and so  $\Delta = \Delta_i^{-\varepsilon}$ .  $\square$

**Lemma 3.7** *The point  $P$  is a vertex of no triangle.*

*Proof.* Let  $i$  and  $\varepsilon$  be as in Lemma 3.6. The point  $P$  clearly cannot be a vertex or an interior point of any triangle not listed in that lemma. If there would exist  $\Delta_h^\eta \neq \Delta_i^{-\varepsilon}$ , then necessarily  $\pi_{rs} \geq 2$  for some  $(r, s) \in \{(3, 1), (2, 1), (2, 3), (1, 3)\}$ , which is a contradiction to Lemma 3.4. Therefore  $P$  cannot be a lateral point either.  $\square$

To finish the proof of Theorem 2.1 it therefore suffices to consider the case when  $P$  is a lateral point of one or more triangles, but a vertex of no triangle. All triangles in which  $P$  is a lateral point are of the form  $\Delta_h^\pm$ ,  $1 \leq h \leq 3$ . From Lemma 3.4 we see that there exist at least two such triangles and that  $P$  is not an interior point of any triangle. Suppose that among the triangles there exists a pair  $(\Delta_h^+, \Delta_h^-)$ . Note that such a pair can be found always when there are at least four triangles. If there were no other triangles beyond the pair, then every sector  $C_{ij}$  would be covered by exactly one triangle. If there were further triangles, then at least one of the sectors listed in Lemma 3.4 would be covered by at least two triangles. Hence no such pair exists.

Suppose now that  $P$  is contained in exactly two triangles. By Lemma 3.4 they have to intersect in one of the sectors  $C_{12}$  and  $C_{32}$ . But then the opposite sector (i. e.  $C_{21}$  or  $C_{23}$ ) is covered by no triangle, and we obtain again a contradiction to Lemma 3.4.

We see that  $P$  must be included in exactly three triangles. We have nine pairs  $(C_{ij}, \Delta_h^\varepsilon)$  with  $C_{ij} \subset \Delta_h^\varepsilon$ . By Lemma 3.4, only  $C_{32}$  and  $C_{12}$  can appear in more than one pair. Since  $9 > 6 + 1 + 1$ , at least one of them has to appear exactly three times. But then the opposite sector lacks covering, and we obtain another contradiction with Lemma 3.4. We have proved that the point  $P$  does not exist, and therefore  $\hat{\Gamma} = \emptyset$ .

## 4 Trigons

In Lemma 4.2 we shall observe that every solution to  $\text{Eq}(T, a)$  can be interpreted, for some  $n \geq 2$ , as a homotopy of  $T(*)$  to  $\mathbb{Z}_n(+)$ . Our goal is to show that this set of homotopies is rich enough to separate any two elements within the same group (the three groups are the rows, the columns and the symbols). This is nearly achieved by Lemma 2.4. To solve the remaining cases we have to consider certain configurations within latin bitrades which we shall call *trigons*. We shall first describe them informally.

Suppose that a bitrade  $S_0$  is derived from a dissection of  $\Sigma$  and that  $\Delta$  is a dissecting triangle. Let us consider another dissection, say of  $\Sigma'$ , that yields a bitrade  $S_1$ . Use the latter dissection as a pattern for how to dissect  $\Delta$ . By identifying vertices of  $\Sigma'$  with vertices of  $\Delta$  we obtain in this way a new dissection of  $\Sigma$ . The dissection determines a new bitrade, say  $T$ , that can be regarded as a superimposition of  $S_0$  and  $S_1$ . The ensuing definition stipulates that the triple representing  $\Delta$  in  $S_0^\Delta$  becomes a trigon in  $T$  (while in  $S_0$  it is not regarded as a trigon).

A *trigon* in a latin bitrade  $T = (T^*, T^\Delta)$  is a triple  $c = (c_1, c_2, c_3)$ ,  $c \notin T^*$ , for which there exist elements  $c'_i \neq c_i$  such that  $(c_1, c_2, c'_3)$ ,  $(c_1, c'_2, c_3)$  and  $(c'_1, c_2, c_3)$  belong to  $T^\Delta$ . These three triples will be called the *corner triples* of  $c$ .

In Lemma 4.3 we shall show how to recombine a homotopy  $\varphi : S_0 \rightarrow \mathbb{Z}_n$  and a homotopy  $\psi : S_1 \rightarrow \mathbb{Z}_m$  into a homotopy  $T \rightarrow \mathbb{Z}_{nm}$ . If  $S_0$  and  $S_1$  are the bitrades based on dissections that have been described above, then one can say that the new homotopy is obtained from  $\varphi$  by embedding  $\mathbb{Z}_n$  into  $\mathbb{Z}_{nm}$  and refining the images of points within  $\Delta$  by means of  $\psi$ . This construction will provide an inductive tool for how to separate an element within a trigon from an element outside. In Lemma 4.4 we shall show that trigons always arise in those cases when Lemma 2.4 does not suffice to separate  $\bar{b}_i$  and  $\bar{a}_i$ . This makes the inductive argument possible, but first we have to show that any trigon  $c$  induces trades  $S_0$  and  $S_1$  that are smaller than  $T$ . In other words, we have to explain how the above construction of  $T$  from  $S_0$  and  $S_1$  can be repeated when  $T$  is not defined by means of dissections.

That is not difficult but it requires a digression that explains how to treat spherical bitrades by means of the combinatorial topology [12, 16, 17]. The

standard way [3, 4] is to associate with a separated latin bitrade  $T = (T^*, T^\Delta)$  a black and white triangulation in which every  $(a_1, a_2, a_3) \in T^*$  is turned into a white triangle  $\{a_1, a_2, a_3\}$  and every  $(b_1, b_2, b_3) \in T^\Delta$  yields a black triangle  $\{b_1, b_2, b_3\}$ . Here we shall use an alternative way how to associate with  $T$  a combinatorial surface. The alternative might be called the *semidual* of the black and white triangulation. It can be described either directly [12, 13], or by the ensuing modification of the standard definition. (Note that if the latin bitrade  $T$  is not separated, then the above definition of the black and white triangles yields only a pseudosurface. The kissing points of such a pseudosurface correspond to elements  $a_t$ ,  $t \in \{1, 2, 3\}$ , that can be divided into two or more rows (columns, symbols) in such a way that the new structure is also a latin bitrade.)

The points of the semidual are the elements of  $T^*$ . They represent the centers of the white triangles. There are two kinds of faces—the *cyclic* faces and the *triangular* faces. Each cyclic face is induced by a (unique) element  $a_t$  that represents a row, a column or a symbol, and is formed by the cyclic sequence of white triangle centers (which we identify with the elements of  $T^*$ ) of those triangles that have  $a_t$  as a vertex. The number of cyclic faces thus equals the aggregate number of rows, columns and symbols. The triangular faces correspond in a one-to-one manner to black faces (and thus also to elements of  $T^\Delta$ ). They are formed by unordered triples that consist of the centers of the three white triangles that are adjacent to the given black triangle. In other words every  $(c_1, c_2, c_3) \in T^\Delta$  induces a triangular face formed by the uniquely determined points  $(c'_1, c_2, c_3)$ ,  $(c_1, c'_2, c_3)$ ,  $(c_1, c_2, c'_3) \in T^*$ .

We shall now define permutations  $\nu_{r,s}$  of  $T^*$  that are dual to the already defined permutations  $\mu_{r,s}$  of  $T^\Delta$ . The cyclic faces are exactly the cycles (in the cyclic decomposition) of  $\nu_{r,s}$ . Assume that  $r, s \in \{1, 2, 3\}$  and that  $r \neq s$ . For  $a = (a_1, a_2, a_3) \in T^*$  consider  $(b_1, b_2, b_3) \in T^\Delta$  and  $a' = (a'_1, a'_2, a'_3) \in T^*$  such that  $a_i \neq b_i$  exactly when  $i = s$  and  $a'_i \neq b_i$  exactly when  $i = r$ . Then  $a' = \nu_{r,s}(a)$ . Choose  $t$  so that  $\{r, s, t\} = \{1, 2, 3\}$ . It is clear that the cycle of  $\nu_{r,s}$  that passes through  $a$  does not change  $a_t$  and coincides with the cyclic face induced by  $a_t$ .

We have  $\nu_{r,s}\nu_{s,t} = \nu_{r,t}$  and  $\nu_{r,s}^{-1} = \nu_{s,r}$ . For every  $j \in \{1, 2, 3\}$  denote  $\nu_{j+1,j-1}$  by  $\tau_j$  (the indices are computed modulo 3). Note that  $\tau_1\tau_2\tau_3 = \tau_2\tau_3\tau_1 = \tau_3\tau_1\tau_2$  is the identity mapping and that the permutations  $\tau_j$  induce an orientation of cyclic faces that can be used to orient the combinatorial surface.

Let now  $T = (T^*, T^\Delta)$  be a separated latin bitrade and let  $c = (c_1, c_2, c_3)$  be a trigon in  $T$ . Let  $\alpha_j \in T^*$  be the triple that agrees with  $c$  in coordinates  $j \pm 1$  and let  $\gamma_j \in T^\Delta$  be the corner triples of  $c$  that also agree with  $c$  in these coordinates.

When we follow the definition of  $\nu_{j\pm 1,j}(\alpha_j)$ , we see that  $\alpha_j$  is first changed to  $\gamma_j$ , and then a change is made in the  $(j \pm 1)$ th coordinate. Hence  $\nu_{j\pm 1,j}(\alpha_j)$  are the vertices, together with  $\alpha_j$ , of the triangular face that is associated with  $\gamma_j$ . They can be expressed as  $\nu_{j-1,j}(\alpha_j) = \tau_{j+1}(\alpha_j)$  and  $\nu_{j+1,j}(\alpha_j) = \tau_{j-1}^{-1}(\alpha_j)$ . In particular,  $\tau_j(\alpha_{j-1})$  and  $\tau_j^{-1}(\alpha_{j+1})$  are vertices of the triangular faces that are induced by  $\gamma_{j-1}$  and  $\gamma_{j+1}$ , respectively.

Let  $\ell_j$  be the length of the cyclic face that is induced by  $c_j$ . Both  $\alpha_{j-1}$  and  $\alpha_{j+1}$  are incident to the cycle, and hence  $\tau_j^{k_j}(\alpha_{j-1}) = \alpha_{j+1}$  for some positive  $k_j < \ell_j$ . We cannot have  $\tau_j(\alpha_{j-1}) = \alpha_{j+1}$  as  $c \neq \gamma_{j-1}$ , and so  $k_j \geq 2$ . This gives us the following characterization of trigons (the converse implication is clear):

**Lemma 4.1** *Let  $T = (T^*, T^\Delta)$  be a separated latin bitrade. A triple  $c = (c_1, c_2, c_3)$  is a trigon in  $T$  if and only if for all  $j \in \{1, 2, 3\}$  there exist  $\alpha_j \in T^*$  that differ from  $c$  exactly in the  $j$ th coordinate, and there exist integers  $k_j$  such that*

$$\tau_j^{k_j}(\alpha_{j-1}) = \alpha_{j+1}, \quad 2 \leq k_j < \ell_j,$$

where  $\ell_j$  is the length of the cycle of  $\tau_j$  that moves  $\alpha_{j\pm 1}$ .

Lemma 4.1 does not require that  $T$  were spherical. However, if it is not, then the oriented closed path (it will be denoted by  $P$ )

$$\alpha_3, \tau_1(\alpha_3), \dots, \tau_1^{k_1}(\alpha_3) = \alpha_2, \tau_3(\alpha_2), \dots, \tau_3^{k_3}(\alpha_2) = \alpha_1, \tau_2(\alpha_1), \dots, \tau_2^{k_2}(\alpha_1) = \alpha_3$$

need not separate the surface into two disjoint parts. Since we are interested in spherical latin bitrades we can assume that the separation takes place.

Put  $\beta_j = \tau_{j-1}^{k_{j-1}-1}(\alpha_{j+1})$  and consider the subpath  $\beta_j, \alpha_j, \tau_{j+1}(\alpha_j)$  of  $P$ . We shall denote it by  $P_j$ ,  $j \in \{1, 2, 3\}$ . We have  $\tau_j^{-1}(\beta_j) = \tau_{j+1}(\alpha_j)$  since

$$\tau_j(\tau_{j+1}(\alpha_j)) = \tau_j \tau_{j+1} \tau_{j-1}(\beta_j) = \beta_j.$$

Let  $h_j$  be the length of the cycle of  $\tau_j$  that moves  $\beta_j$ . Then

$$\beta_j, \tau_j(\beta_j), \dots, \tau_j^{h_j-1}(\beta_j) = \tau_{j+1}(\alpha_j)$$

is an oriented path from  $\beta_j$  to  $\tau_{j+1}(\alpha_j)$  that will be denoted by  $Q_j$ . By substituting  $Q_j$  for  $P_j$  in  $P$ ,  $j \in \{1, 2, 3\}$ , we obtain an oriented closed path that is called the *inner circumference* of the trigon  $c$ .

The points upon the circumference and inside are the *inner points* of  $c$ . The other elements of  $T^*$  are the *outer points* of  $c$ . The outer points include the (vertex) points  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$ .

The triangular faces inside the inner circumference determine a subset of  $T^\Delta$  and we define  $S_1^\Delta$  as the union of this subset with  $\{\gamma_1, \gamma_2, \gamma_3\}$ . As  $S_1^*$  take the union of  $\{c\}$  with the set of all inner points. Then  $S_1 = (S_1^*, S_1^\Delta)$  forms a latin bitrade, and we shall call it the *inner bitrade* of the trigon  $c$ .

Similarly, define  $S_0^*$  as the set of all outer points and  $S_0^\Delta$  as the set obtained by unifying  $\{c\}$  with the set of all elements in  $T^\Delta \setminus \{\gamma_1, \gamma_2, \gamma_3\}$  that determine a triangular face outside the inner circumference. Note that all vertices of every such face are outer points. Hence  $S_0 = (S_0^*, S_0^\Delta)$  forms a bitrade as well, and we shall call it the *outer bitrade* of the trigon  $c$ .

To understand the meaning of  $S_0$  and  $S_1$  topologically is easy. Let  $T$  be a spherical bitrade. The trigon  $c$  determines upon the combinatorial sphere of the



semidual a triangular area that is described by the path  $P$ . The sphere of  $S_0$  is obtained by deleting the inner structure of this area which is now considered as a new triangular face. The sphere of  $S_1$  is obtained by deleting everything outside the area and merging  $\alpha_1$ ,  $\alpha_2$  and  $\alpha_3$  into a single new point. This merge converts the three sides of the triangular area into cyclic faces.

We have seen that if  $T$  is spherical, then both  $S_0$  and  $S_1$  are spherical as well. Of course, this can be also proved formally [14] by counting the faces and using the eulerian characteristic.

This finishes our digression and we return to the program outlined in the beginning of this section. We shall use the conventions of Section 2 established for a situation in which there are fixed an indecomposable spherical latin bitrade  $T = (T^*, T^\Delta)$  and a triple  $a = (a_1, a_2, a_3) \in T^*$ . (For example, if  $b = (b_1, b_2, b_3) \in T^*$ , then  $\bar{b}_j$  denotes the value assigned to  $b_j$  by  $\text{Eq}(T, a)$  for every  $j \in \{1, 2, 3\}$ , and  $\bar{b} = (\bar{b}_2, \bar{b}_1)$ .)

**Lemma 4.2** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade and let  $a = (a_1, a_2, a_3)$  be an element of  $T^*$ . Then there exists an integer  $n \geq 2$  and a homotopy  $\psi = (\psi_1, \psi_2, \psi_3)$  of  $T(*)$  into  $\mathbb{Z}_n$  such that for any  $(b_1, b_2, b_3), (d_1, d_2, d_3) \in T^*$  and  $j \in \{1, 2, 3\}$  the equality  $\psi_j(b_j) = \psi_j(d_j)$  holds if and only if  $\bar{b}_j = \bar{d}_j$  (i.e. if there coincide the values assigned to  $b_j$  and  $d_j$  by the linear system  $\text{Eq}(T, a)$ ).*

*Proof.* Let  $n$  be the least positive integer such that  $n\bar{b}_j \in \mathbb{Z}$  for every  $b = (b_1, b_2, b_3) \in T^* \setminus \{a\}$  and every  $j \in \{1, 2, 3\}$ . Then  $n\bar{b}_1 + n\bar{b}_2 = n\bar{b}_3$ ,  $n\bar{a}_1 + n\bar{a}_2 = 0$  and  $n\bar{a}_3 = n$ . Hence by defining  $\psi = (\psi_1, \psi_2, \psi_3) : T^* \rightarrow \mathbb{Z}_n$  so that  $\psi_j(d_j) \equiv n\bar{d}_j \pmod{n}$  for every  $(d_1, d_2, d_3) \in T^*$  and every  $j \in \{1, 2, 3\}$  we indeed obtain a homotopy. Note that  $0 \leq n\bar{d}_j < n$  if  $j \in \{1, 2\}$  and  $0 < n\bar{d}_3 \leq n$ , by Lemmas 2.3 and 2.4. Hence  $n\bar{b}_j \equiv n\bar{d}_j \pmod{n}$  implies  $\bar{b}_j = \bar{d}_j$ . By Lemma 2.4 none of  $\Delta(c, a)$  equals  $\Sigma$ , and so  $n \geq 2$ .  $\square$

The homotopy  $\psi$  described in the proof of Lemma 4.2 will be called the homotopy *induced* by  $a$ . Denote the integer  $n\bar{d}_j$  by  $\bar{\psi}_j(d_j)$ , for every  $d = (d_1, d_2, d_3) \in T^*$  and every  $j \in \{1, 2, 3\}$ . Then

$$\bar{\psi}_1(d_1) + \bar{\psi}_2(d_2) = \bar{\psi}_3(d_3) \text{ whenever } d \neq a.$$

The triple  $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$  will be called the *near-homotopy* induced by  $a$ . We shall also say that  $n$  is the *width* of  $a$  in  $T$ .

**Lemma 4.3** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade with a trigon  $c = (c_1, c_2, c_3)$ . Let  $S_0$  and  $S_1$  be the outer and inner bitrades of  $c$ , let  $\varphi = (\varphi_1, \varphi_2, \varphi_3)$  be a homotopy  $S_0(*) \rightarrow \mathbb{Z}_m$ , let  $n$  be the width of  $c$  in  $S_1$  and let  $(\bar{\psi}_1, \bar{\psi}_2, \bar{\psi}_3)$  be the near-homotopy induced by  $c \in S_1^*$ . Denote by  $\rho$  the embedding  $\mathbb{Z}_m \rightarrow \mathbb{Z}_{mn}$ ,  $i \mapsto ni$ , put  $h_1 = \rho\varphi_1(c_1)$ ,  $h_2 = \rho\varphi_2(c_2)$  and  $h_3 = h_1 + h_2$ . Finally, choose  $k \in \mathbb{Z}_{mn}$  so that*

$$k \equiv \varphi_3(c_3) - \varphi_1(c_1) - \varphi_2(c_2) \pmod{m}.$$

*For  $j \in \{1, 2, 3\}$  and  $b = (b_1, b_2, b_3) \in T^*$  put  $\varphi'_j(b_j) = \rho\varphi_j(b_j)$  if  $b$  is an outer point of  $c$  and  $\varphi'_j(b_j) = h_j + \bar{\psi}_j(b_j)k$  if  $b$  is an inner point of  $c$ . Then  $\varphi' = (\varphi'_1, \varphi'_2, \varphi'_3)$  is a homotopy  $T(*) \rightarrow \mathbb{Z}_{mn}$ .*

*Proof.* The two formulas that define  $\varphi'_j$  overlap in  $c_j$ . The case  $j \in \{1, 2\}$  is disambiguous since then  $\bar{\psi}_j(c_j) = 0$ . From  $h_3 = h_1 + h_2$  and  $\bar{\psi}_3(c_3) = n$  we obtain that

$$h_3 + \bar{\psi}_3(c_3)k \equiv n(\varphi_1(c_1) + \varphi_2(c_2) + k) \pmod{mn}.$$

Since  $k \equiv \varphi_3(c_3) - \varphi_1(c_1) - \varphi_2(c_2) \pmod{m}$  we see that  $h_3 + \bar{\psi}_3(c_3)k \equiv n\varphi_3(c_3) \pmod{mn}$ , and so  $h_3 + \bar{\psi}_3(c_3)k = \rho\varphi_3(c_3)$ .

The definition of  $\varphi'$  is hence correct and  $\varphi'_1(b_1) + \varphi'_2(b_2)$  clearly equals  $\varphi'_3(b_3)$  if  $b = (b_1, b_2, b_3) \in T^*$  is an outer point. If  $b$  is an inner point, then  $\bar{\psi}_1(b_1) + \bar{\psi}_2(b_2) = \bar{\psi}_3(b_3)$  and so the equality holds as well.  $\square$

Fix an (indecomposable) spherical latin bitrade  $T = (T^*, T^\Delta)$  and a triple  $a = (a_1, a_2, a_3) \in T^*$ . Denote by  $a'_j$ ,  $j \in \{1, 2, 3\}$ , the elements such that the triples  $\eta_1 = (a'_1, a_2, a_3)$ ,  $\eta_2 = (a_1, a'_2, a_3)$  and  $\eta_3 = (a_1, a_2, a'_3)$  belong to  $T^\Delta$ . Note that  $\mu_{j-1, j+1}(\eta_{j+1}) = \eta_{j-1}$  and denote by  $k_j$  the length of the respective cycle. Then  $\mu_{j-1, j+1}^{k_j-1}(\eta_{j-1}) = \eta_{j+1}$  and  $k_j \geq 2$ .

In the proof of Lemma 4.4 some phrases will be expressed in a shortened way. By saying that  $c = (c_1, c_2, c_3) \in T^\Delta$  *degenerates at*  $(s, t) \in \Sigma$  we shall mean that  $\Delta(c, a)$  degenerates and  $(s, t) = (\bar{c}_2, \bar{c}_1)$ . We shall be also saying that an element  $b_j$  *shrinks at*  $(s, t)$  if at  $(s, t)$  there degenerates every  $c = (c_1, c_2, c_3) \in T^\Delta$  with  $c_j = b_j$ . By Lemma 2.4, if  $b_j \neq a_j$  and  $\bar{b}_j = \bar{a}_j$ , then  $b_j$  shrinks at some lateral point of  $\Sigma$ .

**Lemma 4.4** *The triangle  $\Delta(\eta_j, a)$  degenerates for no  $j \in \{1, 2, 3\}$ . Fix  $j \in \{1, 2, 3\}$  and define integers  $0 \leq i_0 < \dots < i_\ell = k_j - 1$  as those  $i$ ,  $0 \leq i < k_j$ , for which  $\Delta(\mu_{j-1, j+1}^i(\eta_{j-1}), a)$  does not degenerate. Put  $\gamma_r = \mu_{j-1, j+1}^{i_r}(\eta_{j-1})$ ,  $0 \leq r \leq \ell$ , and denote by  $b_r^\pm$  the  $(j \pm 1)$ th coordinate of  $\gamma_r$ . For  $0 \leq r < \ell$  define  $\beta_r = (f_1, f_2, f_3)$  so that  $f_j = a_j$ ,  $f_{j-1} = b_r^-$  and  $f_{j+1} = b_{r+1}^+$ . Then either  $\beta_r \in T^*$  and  $i_{r+1} = i_r + 1$ , or  $\beta_r$  is a trigon and  $i_{r+1} > i_r + 1$ . If  $f' = (f'_1, f'_2, f'_3) \in T^\Delta$  agrees with  $\beta_r$  in two coordinates, then  $\Delta(f', a)$  never degenerates.*

*If  $b = (b_1, b_2, b_3) \in T^*$  and  $j \in \{1, 2, 3\}$  are such that  $\bar{b}_j = \bar{a}_j$  and  $b_j \neq a_j$ , then there exists unique  $r$ ,  $0 \leq r < \ell$ , such that  $\beta_r$  is a trigon and  $b$  is an outer point of  $\beta_r$ .*

*Proof.* If  $b \in T^*$  and  $b \neq a$ , then  $\bar{b}$  is never equal to a vertex of  $\Sigma$ , by Lemma 2.4. Therefore no  $c \in T^\Delta$  degenerates at a vertex of  $\Sigma$ , and hence  $\eta_j$  cannot degenerate for any  $j \in \{1, 2, 3\}$ .

Assume now that  $j = 1$ ; the other cases are similar. By the definition,  $\gamma_r = (a_1, b_r^+, b_r^-)$  whenever  $0 \leq r \leq \ell$ . Suppose that  $r < \ell$  and define  $c_2$  and  $c_3$  by  $(a_1, c_2, b_r^-) \in T^*$  and  $(a_1, c_2, c_3) = \mu_{3,2}(\gamma_r) \in T^\Delta$ . Then  $(a_1, b_{r+1}^+, b_r^-) \in T^* \Leftrightarrow c_2 = b_{r+1}^+ \Leftrightarrow (a_1, b_{r+1}^+, c_3) \in T^\Delta \Leftrightarrow \mu_{3,2}(\gamma_r) = \gamma_{r+1}$ . Thus  $\beta_r \in T^*$  if and only if  $i_{r+1} = i_r + 1$ .

All triangles  $\Delta(\gamma_r, a)$  are inside  $\Sigma$ , by Lemma 2.3. One side of such a triangle is upon the horizontal axis. Another side, which is upon the left, is parallel to

the vertical axis and the third side is parallel to the line  $x+y=1$ . The triangles  $\Delta(\gamma_r, a)$  and  $\Delta(\gamma_{r+1}, a)$  have a common vertex, but that obviously is the only nonempty intersection of  $\Delta(\gamma_r, a)$  and  $\Delta(\gamma_s, a)$  whenever  $0 \leq r < s \leq \ell$ . Hence if  $b = (b_1, b_2, b_3) \in T^\Delta \setminus \{a\}$  is such that  $\bar{b} = (h, 0)$  is upon the horizontal axis, then  $\bar{b}$  has to coincide, by Lemmas 2.2 and 2.4, with the common vertex of  $\Delta(\gamma_r, a)$  and  $\Delta(\gamma_{r+1}, a)$ , for some  $r$ ,  $0 \leq r < \ell$ . The value of  $r$  will be now regarded as fixed. Thus  $h = \bar{b}_r^- = \bar{b}_{r+1}^+ = \bar{c}_2$ .

We shall need two auxiliary claims:

### Claim A

Suppose that  $u = (u_1, u_2, b_r^-) \in T^\Delta$  is such that  $u_1$  shrinks at  $(h, 0)$ . Let  $(u'_1, u'_2, b_r^-)$  be equal to  $\mu_{1,2}(u)$ . Then either  $u'_2 = b_{r+1}^+$ , or both  $u'_1$  and  $u'_2$  shrink at  $(h, 0)$ .

The proof of Claim A depends upon

### Claim B

Suppose that  $v = (v_1, u'_2, v_3) \in T^\Delta$  is such that  $v$  degenerates at  $(h, 0)$ . Let  $v' = (v'_1, u'_2, v'_3)$  be equal to  $\mu_{3,1}(v)$ . Then either  $v'$  degenerates at  $(h, 0)$ , or  $v' = \gamma_{r+1}$ .

The assumptions of Claim B imply  $\bar{v}_1 = 0$ ,  $\bar{u}'_2 = \bar{v}_3 = h$  and  $(v'_1, u'_2, v_3) \in T^*$ . Thus  $\bar{v}'_1 = 0$ , and  $v'_1$  shrinks at  $(h, 0)$  if  $v'_1 \neq a_1$ , by Lemma 2.4. Assume  $v'_1 = a_1$  and suppose that  $v'$  does not degenerate. Then  $v'$  has to be equal to one of  $\gamma_s$ ,  $0 \leq s \leq \ell$ , and from  $\bar{u}'_2 = h$  we see that there must be  $s = r+1$ .

With Claim B proved we can turn to Claim A. Assume  $u'_2 \neq b_{r+1}^+$ . We have  $(u_1, u'_2, b_r^-) \in T^*$ , and thus  $u' = (u_1, u'_2, u_3) \in T^\Delta$  for some  $u_3$ . Now,  $(u'_1, u'_2, b_r^-) = \mu_{1,3}(u')$ , and therefore  $\mu_{1,2}(u) = \mu_{3,1}^{k'}(u')$ , where  $k'+1$  is the length of the cycle of  $\mu_{3,1}$  that passes through  $u'$ . The triple  $u'$  degenerates at  $(h, 0)$  since  $u_1$  shrinks at  $(h, 0)$ . No  $\mu_{3,1}^i(u')$  equals  $\gamma_{r+1}$  since  $u'_2 \neq b_{r+1}^+$ . Hence repeated applications of Claim B imply that  $\mu_{3,1}^{k'}(u') = \mu_{1,2}(u)$  degenerates at  $(h, 0)$ . Thus  $\bar{u}'_1 = 0$ , and  $u'_1$  shrinks at  $(h, 0)$  if  $u'_1 \neq a_1$ , by Lemma 2.4. To finish the proof of Claim A it thus suffices to show how to deduce a contradiction from the assumption that  $u'_1 = a_1$ . However, in such a case  $\mu_{1,2}(u) = \gamma_r$ . That cannot be since  $\mu_{1,2}(u)$  degenerates while  $\gamma_r$  does not.

Suppose now that  $\beta_r \in T^*$ . Then there exists a unique  $c_1 \neq a_1$  such that  $c = (c_1, b_{r+1}^+, b_r^-) \in T^\Delta$ . Note that  $\gamma_r = \mu_{2,1}(c)$  and  $\gamma_r = \mu_{1,2}^k(c)$ , where  $k+1$  is the length of cycle of  $\mu_{1,2}$  that passes through  $c$ . If  $1 \leq i \leq k$  and  $(u'_1, u'_2, b_r^-) = \mu_{1,2}^i(c)$ , then  $u'_2 \neq b_{r+1}^+$ . If  $c$  degenerates, then  $c_1$  shrinks, by Lemma 2.4, and repeated applications of Claim A imply that  $\gamma_r = \mu_{1,2}^k(c)$  degenerates as well. This is a contradiction, and hence  $c$  does not degenerate.

Let us turn to the case  $\beta_r \notin T^*$ . Then  $(a_1, c_2, c_3) = \mu_{3,2}(\gamma_r)$  degenerates at  $(h, 0)$  and  $c_2 \neq b_{r+1}^+$ . Repeated applications of Claim B imply that  $c_2$  shrinks at  $(h, 0)$ . We have  $(a_1, c_2, b_r^-) \in T^*$ , and so there exists  $c'_1$  such that  $(c'_1, c_2, b_r^-) \in T^\Delta$  and  $c'_1 \neq a_1$ . This triple degenerates at  $(h, 0)$  since  $c_2$  shrinks. Lemma 2.4 implies that  $c'_1$  shrinks at  $(h, 0)$  as well. Apply now Claim A repeatedly, starting

from  $(c'_1, c_2, b_r^-)$ . Since  $b_r^-$  does not shrink, we have to hit  $b_{r+1}^+$  at some stage. Therefore there exist  $c'_1$  and  $c_1$  such that  $c'_1$  shrinks at  $(h, 0)$ ,  $(c'_1, b_{r+1}^+, b_r^-) \in T^*$ , and  $c = (c_1, b_{r+1}^+, b_r^-) \in T^\Delta$ . We see that  $\beta_r$  is a trigon with corner triples equal to  $c$ ,  $\gamma_r$  and  $\gamma_{r+1}$ .

Our next step is to show that  $c$  does not degenerate. The proof is practically the same as in the case  $\beta_r \in T^*$ . Assume that  $c$  degenerates and apply Claim A repeatedly. The permutation  $\mu_{1,2}$  brings  $c$  to  $\gamma_r$  before one hits  $b_{r+1}^+$ , and so  $\gamma_r$  has to degenerate. This is a contradiction and hence  $c$  cannot degenerate.

To finish the proof of the lemma it thus remains to show that  $b$  is an outside point of the trigon  $\beta_r$  whenever  $b = (b_1, b_2, b_3) \in T^*$  is such that  $\bar{b} = (h, 0)$  and  $b_1 \neq a_1$ . We shall be thus also proving that no such  $b$  exists when  $\beta_r \in T^*$ .

Let  $B^* \subseteq T^*$  be the set of all counterexamples (for a fixed  $h$ ) and assume that  $B^* \neq \emptyset$ . Set  $B = \{b_1; (b_1, b_2, b_3) \in B^* \text{ for some } b_2 \text{ and } b_3\}$  and note that if  $b' = (b_1, b'_2, b'_3) \in T^*$  and  $b_1 \in B$ , then  $\bar{b}'$  equals  $(h, 0)$  since  $b_1$  shrinks at  $(h, 0)$  by Lemma 2.4. If  $\beta_r$  is a trigon, then for a given  $b_1 \neq a_1$  either all elements  $b'$  are among the inside points of the trigon or all such elements are outside points. Since we assume that  $(b_1, b_2, b_3)$  is an inside point for some  $b_2$  and  $b_3$ , the former alternative has to take place. It follows that  $B^* = \{(b_1, b_2, b_3) \in T^*; b_1 \in B\}$ . Put  $B^\Delta = \{(b_1, b_2, b_3) \in T^\Delta; b_1 \in B\}$ . We shall show that  $(B^*, B^\Delta)$  is a latin bitrade. This will yield the sought contradiction since  $T$  is assumed to be indecomposable.

Obviously it suffices to prove

### Claim C

If  $(e_1, e_2, e_3) \in B^\Delta$  and  $(d_1, e_2, e_3) \in T^*$ , then  $d_1 \in B$ ; and

### Claim D

If  $(d_1, d_2, d_3) \in B^*$  and  $(e_1, d_2, d_3) \in T^\Delta$ , then  $e_1 \in B$ .

We shall first explain how Claim C implies Claim D. Suppose that  $(d_1, d_2, d_3) \in B^*$  and that  $e = (e_1, d_2, d_3) \in T^\Delta$ . Let  $d'_3$  be such that  $d = (d_1, d_2, d'_3) \in T^\Delta$ . Then  $e = \mu_{1,3}(d)$  and  $d \in B^\Delta$ . By working inductively along  $\mu_{3,1} = \mu_{1,3}^{-1}$  it is therefore enough to prove that if  $d' = (d'_1, d_2, d'_3) \in B^\Delta$ , then  $\mu_{3,1}(d') = (d''_1, d_2, d''_3)$  belongs to  $B^\Delta$  as well. However, this is clear from Claim C since  $(d'_1, d_2, d'_3) \in T^*$ .

To prove Claim C put  $e = (e_1, e_2, e_3)$  and  $d = (d_1, e_2, e_3)$ . We have  $e_1 \in B$ , and hence  $e_1$  shrinks at  $(h, 0)$ , by Lemma 2.4. Therefore  $e$  degenerates at  $(h, 0)$  and  $\bar{d}$  equals  $(h, 0)$ .

If  $\beta_r$  is a trigon, consider  $d_2$  and  $d_3$  such that  $(e_1, d_2, e_3) \in T^*$  and  $(e_1, e_2, d_3) \in T^*$ . Both these triples belong to  $B^*$  and are inner points of  $\beta_r$  since  $e_1 \neq a_1$ . The third triple from  $T^*$  that is induced by  $e = (e_1, e_2, e_3)$  is also an inner point of  $\beta_r$ , unless  $e$  is the corner triple. The third triple is equal to  $d$ , of course. We have proved above that no corner triple of  $\beta_r$  degenerates, and therefore  $d$  has to be an inner point of  $\beta_r$ . The trigon  $\beta_r$  has been constructed in such a way that  $\gamma_r$  and  $\gamma_{r+1}$  are its corner triples and that any  $u = (a_1, u_2, u_3) \in T^*$  with

$\bar{u} = (h, 0)$  is an outer point of  $\beta_r$ . Since  $d$  is an inner point with  $\bar{d} = (h, 0)$ , there cannot be  $d_1 = a_1$ . Hence  $d_1 \in B$ .

If  $\beta_r \in T^*$ , then the argument is similar. From Lemma 2.4 we deduce that  $d_1 \in B$  or that  $d_1 = a_1$ . If  $d_1 = a_1$ , then  $d = \beta_r$  since here we assume the existence of only one  $u = (a_1, u_2, u_3)$  such that  $\bar{u} = (h, 0)$ . If  $d = \beta_r$ , then  $e$  is equal to the triple  $c = (c_1, b_{r+1}^+, b_r^-) \in T^\Delta$ . We have already proved above that this triple does not degenerate.

The proof is finished.  $\square$

**Theorem 4.5** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade. If  $a = (a_1, a_2, a_3) \in T^*$ ,  $b = (b_1, b_2, b_3) \in T^*$  and  $i \in \{1, 2, 3\}$  are such that  $a_i \neq b_i$ , then there exist  $n \geq 2$  and a homotopy  $\varphi = (\varphi_1, \varphi_2, \varphi_3) : T(*) \rightarrow \mathbb{Z}_n(+)$  such that  $\varphi_i(a_i) \neq \varphi_i(b_i)$ .*

*Proof.* Consider first the homotopy  $\psi = (\psi_1, \psi_2, \psi_3)$  induced by  $a$ . If  $\psi_i(a_i) = \psi_i(b_i)$ , then  $\bar{a}_i = \bar{b}_i$  by Lemma 2.4, and  $b$  is an outer point of a trigon  $c = (c_1, c_2, c_3)$  in which  $c_i = a_i$ , by Lemma 4.4.

In the rest we shall proceed by induction on the size of  $T$ . Let  $S_0$  be the outer trade of  $c$ , and let  $S_1$  be the inner trade. Recall that  $c \in S_0^\Delta$ . By induction assumption there exist  $m \geq 2$  and a homotopy  $\varphi = (\varphi_1, \varphi_2, \varphi_3) : S_0(*) \rightarrow \mathbb{Z}_m$  such that  $\varphi_i(a_i) \neq \varphi_i(b_i)$ . Lemma 4.3 describes a procedure how to find  $n' > n$  and a homotopy  $\varphi' = (\varphi'_1, \varphi'_2, \varphi'_3) : T(*) \rightarrow \mathbb{Z}_{n'}$  such that  $\varphi'_i(a_i) \neq \varphi'_i(b_i)$ .  $\square$

**Corollary 4.6** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade. Then both  $T(*)$  and  $T(\Delta)$  can be embedded into a finite abelian group.*

*Proof.* By Theorem 4.5 for every triple  $(a, b, j) \in T^* \times T^* \times \{1, 2, 3\}$  such that  $a_j \neq b_j$  there exists an integer  $n = n[a, b, j]$  and a homotopy  $\sigma = \sigma[a, b, j] : T(*) \rightarrow \mathbb{Z}_n$  in which  $\sigma_j(a_j) \neq \sigma_j(b_j)$ . Put  $G = \prod \mathbb{Z}_{n[a, b, j]}$  and define a homotopy  $\tau : T(*) \rightarrow G$  so that the projection of  $\tau$  to  $\mathbb{Z}_{n[a, b, j]}$  coincides with  $\sigma[a, b, j]$ . There are only finitely many triples  $(a, b, j)$ , and that makes  $G$  finite. The homotopy  $\tau$  differentiates between any two different triples of  $T^*$  and hence it really embeds  $T(*)$  into  $G$ .  $\square$

## 5 Modifications to abelian groups

Suppose that  $T = (T^*, T^\Delta)$  is a spherical latin bitrade.  $T$  is assumed to be indecomposable and we define  $m = o_1 + o_2 + o_3$  in the same way as in Section 2. The linear system  $\text{Eq}(T)$  has  $m$  variables and  $s = m - 2$  equalities. Relabel the variables by  $x_1, \dots, x_m$  in such a way that the unknowns  $x_j$  correspond to rows, columns and symbols if  $1 \leq j \leq o_1$ ,  $o_1 < j \leq o_1 + o_2$  and  $o_1 + o_2 < j \leq m$ , respectively. Order the equalities of  $\text{Eq}(T)$  in an arbitrary way and define a

matrix  $B$  so that the  $i$ th row expresses the  $i$ th equation. (If  $x_r + x_s - x_t = 0$  is the equation, then  $b_{ir} = b_{is} = 1$ ,  $b_{it} = -1$ , and  $b_{ij} = 0$  in other cases.)

Denote by  $B_{ij}$  the matrix that is derived from  $B$  by omitting the  $i$ th and  $j$ th column. By [10, Lemma 3.3] the following statement holds. We include the proof since [10] uses an approach that is unnecessarily general for the needs of this paper.

**Lemma 5.1** *Suppose that either  $1 \leq i \leq o_1 < j \leq m$  or  $o_1 < i \leq o_1 + o_2 < j \leq m$ . Then  $|\det B_{ij}| = |\det B_{1m}|$ .*

*Proof.* Let  $C$  be a  $(m+1) \times m$  matrix with rows  $\mathbf{c}_1, \dots, \mathbf{c}_{m+1}$  and let  $\lambda_1, \dots, \lambda_{m+1}$  be coefficients such that  $\sum \lambda_h \mathbf{c}_h = 0$ . Denote by  $C_h$  the square matrix that is obtained from  $C$  by deleting the row  $\mathbf{c}_h$ . Consider  $C_u$  and  $C_v$  where  $1 \leq u < v \leq m+1$ , and denote by  $D$  the matrix obtained from  $C_v$  when the  $u$ th row  $\mathbf{c}_u$  is replaced by  $-\lambda_u \mathbf{c}_v = \sum_{h \neq v} \lambda_h \mathbf{c}_h$ . Then  $\det D = \lambda_u \det C_u = -\lambda_v (-1)^{v-1-u} \det C_v$ , and so  $|\lambda_u \det C_u| = |\lambda_v \det C_v|$ .

Suppose now that  $1 \leq r \leq o_1 < s \leq o_1 + o_2 < t \leq m$ , and form a  $(m+1) \times m$  matrix  $C$  from  $B$  by adding  $(m-1)$ th,  $m$ th and  $(m+1)$ th row such that each of them contains  $m-1$  zeros, and the cell in the column  $r$  (or  $s$ , or  $t$ ) contains the value 1, respectively. Finally, multiply the last row by  $-1$ . Our intention is to show that  $|\det C_u| = |\det C_v|$  when  $m-1 \leq u < v \leq m+1$ .

The determinants vanish when  $B$  is not of rank  $m-2$ , and so we can assume that it is of the full rank. Denote by  $W$  the space of vectors  $(w_1, \dots, w_m)$  in which  $w_1 = \dots = w_{o_1}$ ,  $w_{o_1+1} = \dots = w_{o_1+o_2}$ ,  $w_{o_1+o_2+1} = \dots = w_m$  and  $w_1 + w_{o_1+1} - w_{o_1+o_2+1} = 0$ . Let  $\mathbf{w}_1$  (or  $\mathbf{w}_2$ , or  $\mathbf{w}_3$ ) be the element of  $W$  such that  $w_1 = 1$ ,  $w_{o_1+1} = -1$  and  $w_{o_1+o_2+1} = 0$  (or  $w_1 = 0$  and  $w_{o_1+1} = 1 = w_{o_1+o_2+1}$ , or  $w_{o_1+1} = 0$  and  $w_1 = 1 = w_{o_1+o_2+1}$ , respectively). The rows  $\mathbf{c}_1, \dots, \mathbf{c}_{m-2}$  are orthogonal to the elements of  $W$ . Choose  $\lambda_1, \dots, \lambda_{m+1}$  such that not all of them are zero and  $\sum \lambda_h \mathbf{c}_h = 0$ . The scalar product of  $\sum \lambda_h \mathbf{c}_h$  with any element of  $W$  thus vanishes, and hence  $\mathbf{u} = \lambda_{m-1} \mathbf{c}_{m-1} + \lambda_m \mathbf{c}_m + \lambda_{m+1} \mathbf{c}_{m+1}$  is also orthogonal to  $W$ . The scalar products of  $\mathbf{u}$  with  $\mathbf{w}_1$ ,  $\mathbf{w}_2$  and  $\mathbf{w}_3$  yield  $\lambda_{m-1} - \lambda_m = \lambda_m - \lambda_{m+1} = \lambda_{m-1} - \lambda_{m+1} = 0$ . We cannot have  $\lambda = \lambda_{m+1} = \lambda_m = \lambda_{m-1} = 0$  since  $B$  is of rank  $m-2$ . Therefore  $\lambda \neq 0$  and thus  $|\det C_{m-1}| = |\det C_m| = |\det C_{m+1}|$ , by the first part of the proof.

Choose now  $(r, s, t)$  in such a way that  $(i, j)$  is one of  $(r, s)$ ,  $(r, t)$  and  $(s, t)$ , and use the obvious equalities  $|\det C_{m-1}| = |\det B_{st}|$ ,  $|\det C_m| = |\det B_{rt}|$  and  $|\det C_{m+1}| = |\det B_{rs}|$ . The rest is clear since we can move in at most three steps from  $(i, j)$  to any other  $(i', j')$ , including the pair  $(1, m)$ .  $\square$

We shall also need a result that follows from [9, Lemma 3.1]. We give a full proof since [9] seems to be difficult to read. The proof is similar to the proofs of Sections 2. The homotopies described in Lemma 5.2 will be called *trivial*.

**Lemma 5.2** *Let  $(\sigma_1, \sigma_2, \sigma_3)$  be a homotopy of  $T(*)$  into the additive group of integers  $\mathbb{Z}(+)$ . Then  $\sigma_i(a_i) = \sigma_i(b_i)$  for all  $(a_1, a_2, a_3), (b_1, b_2, b_3) \in T^*$  and every  $i \in \{1, 2, 3\}$ .*

*Proof.* Suppose that  $(\sigma_1, \sigma_2, \sigma_3)$  is a nontrivial homotopy. Say that  $c = (c_1, c_2, c_3) \in T^\Delta$  degenerates if  $\sigma_1(c_1) + \sigma_2(c_2) = \sigma_3(c_3)$ . Put  $M = \{(\sigma_1(a_1), \sigma_2(a_2), \sigma_3(a_3)); (a_1, a_2, a_3) \in T^*\}$ ,  $h_3 = \max\{r_3; (r_1, r_2, r_3) \in M\}$ ,  $h_2 = \min\{r_2; (r_1, r_2, h_3) \in M\}$ ,  $h_1 = h_3 - h_2$ ,  $X = \{(a_1, a_2, a_3) \in T^*; \sigma_i(a_i) = h_i, 1 \leq i \leq 3\}$ , and for each  $i \in \{1, 2, 3\}$  define  $Y_i$  as  $\{(c_1, c_2, c_3) \in T^\Delta; \sigma_j(a_j) = h_j \text{ if } 1 \leq j \leq 3 \text{ and } i \neq j\}$ . For each element of  $Y_i$  there exists a unique element of  $X$  that agrees in two coordinates and disagrees in the  $i$ th coordinate. An element  $(c_1, c_2, c_3) \in Y_i$  degenerates if and only if  $\sigma_i(c_i) = h_i$ , and therefore  $D = Y_1 \cap Y_2 \cap Y_3 = Y_1 \cap Y_2 = Y_1 \cap Y_3 = Y_2 \cap Y_3$  consists of exactly those  $c \in Y_1 \cup Y_2 \cup Y_3$  that degenerate. We have  $|X| = |Y_i|$  for all  $i$ , and hence either  $D = Y_1 = Y_2 = Y_3$ , or  $D$  is a proper subset of each  $Y_i$ ,  $1 \leq i \leq 3$ . In the former case all of the mappings  $\mu_{r,s}$  permute  $D$ , and that makes  $(X, D)$  a subtrade. An indecomposable trade contains no proper subtrade, and hence  $(X, D) = (T^*, T^\Delta)$ . However, this is not possible since  $(\sigma_1, \sigma_2, \sigma_3)$  is assumed to be nontrivial.

If  $\mu_{r,s}(D) = D$ , then there exists  $c \in Y_i$  with  $\mu_{r,s}(c) \notin D$  since  $D \subsetneq Y_i$ . This inclusion guarantees the existence of  $c \in Y_i$  with  $\mu_{r,s}(c) \notin D$  also in the case when  $\mu_{r,s}(D) \neq D$ .

Fix  $c = (c_1, c_2, c_3) \in Y_1$  such that  $\mu_{2,1}(c) = c' = (a_1, a_2, c_3) \notin D$ . Note that  $(a_1, c_2, c_3) \in X$ ,  $\sigma_1(a_1) = h_1$ ,  $\sigma_2(c_2) = h_2$  and  $\sigma_3(c_3) = h_3$ . Since  $c'$  does not degenerate, there must be  $\sigma_2(a_2) \neq h_2$ .

There exist elements  $a'_1$  and  $c'_3$  such that both  $(a_1, a_2, c'_3)$  and  $(a'_1, a_2, c_3)$  belong to  $T^*$ . We have  $\sigma_3(c'_3) \neq h_3$  since  $\sigma_1(a_1) + \sigma_2(a_2) \neq h_1 + h_2 = h_3$ . Therefore  $\sigma_3(c'_3) = h_1 + \sigma_2(a_2) < h_3$ , and thus  $\sigma_2(a_2) < h_2$ . On the other hand  $\sigma_1(a'_1) + \sigma_2(a_2) = h_3$  implies  $\sigma_2(a_2) \geq h_2$ , by the definition of  $h_2$ . We have obtained a contradiction.  $\square$

Let  $G = G(+)$  be an abelian group. Consider, for a while,  $\text{Eq}(T)$  as a set of equations in  $G$ . We can regard each solution as a triple  $(\sigma_1, \sigma_2, \sigma_3)$  of mappings into  $G$  such that  $\sigma_1(a_1) + \sigma_2(a_2) = \sigma_3(a_3)$  for all  $(a_1, a_2, a_3) \in T^*$ . The solutions thus correspond to homotopies  $T(*) \rightarrow G$ . Trivial homotopies can be obtained easily by choosing elements  $g$  and  $h$  of  $G$  and by setting  $\sigma_1(a_1) = g$ ,  $\sigma_2(a_2) = h$  and  $\sigma_3(a_3) = g + h$ , for all  $(a_1, a_2, a_3) \in T^*$ .

Consider  $a = (a_1, a_2, a_3) \in T^*$  and assume that  $a_1$ ,  $a_2$  and  $a_3$  have been relabelled as  $x_r$ ,  $x_s$  and  $x_t$ , respectively. Thus  $1 \leq r \leq o_1 < s \leq o_1 + o_2 < t \leq m$ . If  $\det B_{rs} = 0$ , then there exists a nonzero integer vector  $v$  such that  $Bv^\top = 0$  and  $v_r = v_s = v_t = 0$ . Such a vector supplies a solution to  $\text{Eq}(T)$  in  $\mathbb{Z}$ , and thus it yields a nontrivial homotopy  $T(*) \rightarrow \mathbb{Z}$ . However, no such homotopy exists, by Lemma 5.2, and hence  $\det B_{rs} \neq 0$ . Remove now from  $B$  the columns  $r$ ,  $s$ ,  $t$  and the row that corresponds to the equation  $x_r + x_s = x_t$ . Let it be the  $i$ th row. The new matrix, say  $C$ , can be thus obtained from  $B_{rs}$  by deleting a column and the  $i$ th row. Since this row contains a single nonzero value, and since this value is equal to  $\pm 1$  and is in the column that is being deleted we see that  $|\det C| = |\det B_{rs}| \neq 0$ . The matrix  $C$  is the matrix of the linear system  $\text{Eq}(T, a)$ . We have hence proved the following statement:



**Lemma 5.3** *The system of linear equations  $\text{Eq}(T, a)$  has a unique solution in rational numbers. Furthermore,  $\det B_{1m} \neq 0$ .*

Note that results of Sections 2 and 4 assume the validity of Lemma 5.3. Thus only at this point we can regard as proved the fact that each spherical latin bitrade can be embedded into a finite abelian group. We shall now take a more systematic look upon such embeddings and connect Corollary 4.6 to earlier results of [8] in which there was developed a machinery of group modifications. We shall limit our discussion only to aspects relevant to abelian groups, and refer to [8] for further ramifications. The terminology of [8] is somewhat different, but that should not cause difficulties.

We will view homotopies as morphisms in the category of partial quasigroups. Both groups and latin bitrades can be regarded as subcategories of this category: A group homomorphism  $f : G \rightarrow H$  is identified with a homotopy  $(f, f, f)$ , and a homotopy  $(\sigma_1, \sigma_2, \sigma_3)$  of  $T(*) \rightarrow S(*)$  is regarded (for our purposes here) as a morphism  $(T^*, T^\Delta) \rightarrow (S^*, S^\Delta)$ .

The following statement is clear and does not require a proof. The notation  $\sigma[a]$  that is introduced in Lemma 5.4 will be used further on (in particular, in and before Lemma 5.6).

**Lemma 5.4** *Let  $T = (T^*, T^\Delta)$  be a latin bitrade and let  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  be a homotopy of  $T(*)$  into an abelian group  $K = K(+)$ . For  $a = (a_1, a_2, a_3) \in T^*$  define a triple of mappings  $\sigma[a] = (\tau_1, \tau_2, \tau_3)$  in such a way that  $\tau_i(b_i) = \sigma_i(b_i) - \sigma_i(a_i)$  for all  $i \in \{1, 2, 3\}$  and all  $(b_1, b_2, b_3) \in T^*$ . Then  $\sigma[a]$  is also a homotopy of  $T(*)$  into  $K$  and  $\tau_i(a_i) = 0$  for each  $i \in \{1, 2, 3\}$ . The homotopy  $\sigma$  is an embedding if and only if  $\sigma[a]$  is an embedding, and  $\sigma$  is trivial if and only if  $\sigma[a]$  is trivial.*

By a *modification* (or *reflexion*) of a category into a subcategory one understands morphisms  $g_K : K \rightarrow G(K)$  that are defined for each object  $K$ . The object  $G(K)$  is in the subcategory and for all morphisms  $h : K \rightarrow H$  where the target  $H$  is in the subcategory there exists (in the subcategory) a unique morphism  $k : G(K) \rightarrow H$  such that  $h = kg_K$ . As a typical examples one can take the natural projections  $G \rightarrow G/G'$  which yield a modification from the category of groups into the subcategory of abelian subgroups.

Let  $T = (T^*, T^\Delta)$  be a latin bitrade. For every  $(a_1, a_2, a_3) \in T^*$  regard again elements  $a_i$  as variables and denote by  $F = F(T)$  the free abelian group generated by them. The group  $F$  is thus of rank  $m = o_1 + o_2 + o_3$ , but we do not require that the size of  $T$  is necessarily equal to  $m - 2$ . Regard now  $\text{Eq}(T)$  as a set of elements  $a_1 + a_2 - a_3 \in F$ , denote by  $N(T)$  the subgroup generated by these elements, and denote by  $\mathbf{G}(T)$  the factor-group  $F(T)/N(T)$ . Define  $\mathbf{g}_T$  as a triple of mappings  $(g_1, g_2, g_3)$  such that  $g_i(a_i) = a_i + N(T)$ . Then  $\mathbf{g}_T$  is clearly a homotopy of  $T(*)$  into  $\mathbf{G}(T)$ , and it can be verified easily that  $\mathbf{g}_T : T \rightarrow \mathbf{G}(T)$  defines a modification from the category of latin bitrades to the category of abelian groups. This also follows immediately from [8, Proposition 3.1] since  $\mathbf{G}(T)$  can be identified with  $G(T(*))/(G(T(*)))'$  (by  $G$  one denotes in [8] the modification into the category of all groups).



**Lemma 5.5** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade. Then  $\mathbf{g}_T$  provides an embedding of  $T(*)$  into  $\mathbf{G}(T)$ .*

*Proof.* Let  $a = (a_1, a_2, a_3) \in T^*$ ,  $b = (b_1, b_2, b_3) \in T^*$  and  $i \in \{1, 2, 3\}$  be such that  $a_i \neq b_i$ . Put  $\mathbf{g}_T = (g_1, g_2, g_3)$ . Our goal is to prove that  $g_i(a_i) \neq g_i(b_i)$ . By Theorem 4.5 there exists an abelian group  $K$  and a homotopy  $(\sigma_1, \sigma_2, \sigma_3)$  of  $T(*)$  into  $K$  such that  $\sigma_i(a_i) \neq \sigma_i(b_i)$ . Because  $\mathbf{g}_T$  is a modification, there must exist a group homomorphism  $\varphi : \mathbf{G}(T) \rightarrow K$  such that  $\sigma_j = \varphi g_j$  for all  $j \in \{1, 2, 3\}$ . But that means  $g_i(a_i) \neq g_i(b_i)$ .  $\square$

Let again  $T = (T^*, T^\Delta)$  be a (general) latin bitrade. Put  $\mathbf{g}_T = (g_1, g_2, g_3)$ . Following [8] define  $\mathbf{H}(T)$  as the subgroup of  $\mathbf{G}(T)$  generated by the set of all  $g_i(b_i) - g_i(a_i)$ , where  $(a_1, a_2, a_3), (b_1, b_2, b_3) \in T^*$  and  $i \in \{1, 2, 3\}$ . Note that to generate  $\mathbf{H}(T)$  we may consider only  $i \in \{1, 2\}$ , and that we also may keep  $a = (a_1, a_2, a_3)$  fixed if  $b = (b_1, b_2, b_3)$  runs through  $T^*$  (to see the latter note that  $g_i(b_i) - g_i(c_i) = (g_i(b_i) - g_i(a_i)) - (g_i(c_i) - g_i(a_i))$ ). Put  $\mathbf{g}_T[a] = (h_1, h_2, h_3)$ . From Lemma 5.4 we immediately obtain the following observation:

**Lemma 5.6** *The triple  $\mathbf{g}_T[a] = (h_1, h_2, h_3)$  is a homotopy from  $T(*)$  to  $\mathbf{H}(T)$ , and  $\mathbf{g}_T[a]$  embeds  $T(*)$  into  $\mathbf{H}(T)$  if and only  $\mathbf{g}_T$  embeds  $T(*)$  into  $\mathbf{G}(T)$ . Furthermore,  $\mathbf{H}(T)$  is generated by the set  $\{h_1(b_1), h_2(b_2); (b_1, b_2, b_3) \in T^*\}$ .*

**Theorem 5.7** *Let  $T = (T^*, T^\Delta)$  be a spherical latin bitrade. Then  $T(*)$  can be embedded into the abelian group  $\mathbf{H}(T)$  and this group is finite.*

*Proof.* The fact that  $T(*)$  embeds into  $\mathbf{H}(T)$  follows immediately from Lemmas 5.5 and 5.6. The group  $\mathbf{H}(T)$  has only finitely many generators and so it suffices to show that it is a torsion group. Choose  $a \in T^*$  and assume the contrary. Then there exists a surjective group homomorphism  $\pi : \mathbf{H}(T) \rightarrow \mathbb{Z}$ , and from Lemma 5.6 we see that  $(\pi h_1, \pi h_2, \pi h_3)$  is a homotopy  $T(*) \rightarrow \mathbb{Z}$  such that  $\mathbb{Z}$  is generated by the set of all  $\pi h_1(b_1)$  and  $\pi h_2(b_2)$ , where  $b = (b_1, b_2, b_3)$  runs through  $T^*$ . Since  $\pi h_i(a_i) = 0$  for all  $i \in \{1, 2, 3\}$  we also see that the homotopy  $(\pi h_1, \pi h_2, \pi h_3)$  is not trivial. However, that contradicts Lemma 5.2.  $\square$

Let us mention that not all spherical latin bitrades embed into a cyclic group—the least known example has size 12 and is mentioned already in [7]. Ian Wanless found in 2006 several latin bitrades that can be embedded into no group and are of size 24 and genus 4. The following bitrade is toroidal and also embeds into no group. It has 5 rows, 6 columns, 7 symbols, and is of size 18.

$*$	$f$	$a$	$b$	$c$	$d$	$g$
$e$	1		3	4	5	
$x$	3	6	2	5		7
$y$	5				1	
$z$			4	2		
$t$	7	5	6	3		2

$\Delta$	$f$	$a$	$b$	$c$	$d$	$g$
$e$	3		4	5	1	
$x$	7	5	6	3		2
$y$	1				5	
$z$			2	4		
$t$	5	6	3	2		7

Denote the left latin trade as  $T(*)$  and the right trade  $T(\triangle)$ . It is not difficult to see that  $\mathbf{H}(T(\triangle)) \cong \mathbb{Z}_{10}$  and that  $T(\triangle)$  embeds into  $\mathbf{H}(T(\triangle))$ . In fact  $H(T(\triangle))$ , where  $H$  means (as in [8]) the noncommutative version of  $\mathbf{H}$ , is isomorphic to  $\mathbb{Z}_{10}$  as well. Our claim that the bitrade embeds into no group refers to  $T(*)$ . To see that it embeds into no abelian group is easy, but the general case seems to deserve a formal proof:

**Lemma 5.8** *All homotopies of  $T(*)$  into a group are trivial.*

*Proof.* The rows are denoted by  $e, x, y, z$  and  $t$ , and the columns by  $f, a, b, c, d$  and  $g$ . We shall regard these elements as generators of a group  $G$  to which there exists a homotopy from  $T(*)$ . If the homotopy is nontrivial, then there exists a nontrivial homotopy in which  $e = 1 = f$ . To see that modify Lemma 5.5 so that it is valid for noncommutative groups as well ([9, Lemma 1.6] or [8, Lemma 3.3]). Assuming  $e = f = 1$  we get  $b = eb = xf = x$ ,  $c = ec = zb = zx$ ,  $d = ed = yf = y = xc = xzx$  and  $zc = xb = x^2$ . Now,  $c = zx$  yields  $z^2x = x^2$ , and we obtain  $x = z^2$ . Using  $t = xc^{-1}$  and  $g = x^{-1}t$  we find that all generators are powers of  $z$ . In particular,  $e = 1$ ,  $x = z^2$ ,  $y = z^5$ ,  $t = z^{-1}$ ,  $f = 1$ ,  $b = z^2$ ,  $c = z^3$ ,  $d = z^5$ ,  $g = z^{-3}$  and  $a = t^{-1}y = z^6$ . Now,  $z = tb = xa = z^8$  yields  $z^7 = 1$ , and so  $1 = ef = y^2 = z^{10}$  implies  $z = 1$ . Group  $G$  is thus trivial, and so there exists no nontrivial homotopy  $T(*) \rightarrow G$ .  $\square$

The question whether it is possible to embed every spherical latin bitrade into an abelian group got certain publicity at the workshop “Algebraic and geometric aspects of latin trades” that was organized at Charles University, Prague, in February 2006. After submitting the first version of this paper we contacted Cavenagh and Wanless since we knew that they had been working upon the problem. It turned out that they found (amongst others) another proof [5]. Both research efforts have been independent. There are several common features, but there are also quite a few dissimilarities—e.g. [5] does not use dissections.

We finish by two problems. Let  $T = (T^*, T^\triangle)$  be a spherical latin bitrade.

- (1) Is it always possible to retrieve  $T$  from a dissection if  $\mathbf{H}(T)$  is cyclic?
- (2) Must  $\mathbf{H}(T)$  be cyclic when  $T$  can be derived from a separated dissection?

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## References

- [1] V. Batagelj: An improved inductive definition of two restricted classes of triangulations of the plane. *Combinatorics and graph theory (Warsaw 1987)*, 11–18, Banach Center Publ., 25, PWN, Warsaw, 1989.

- [2] G. Brinkman and B. D. McKay: *Guide to using plantri*, version 4.1  
<http://cs.anu.edu.au/~bdm/plantri/>.
- [3] N. J. Cavenagh: The theory and applications of latin bitrades: a survey, *Mathematica Slovaca* **58** (2008), 691–718.
- [4] N. J. Cavenagh and P. Lisoněk: Planar eulerian triangulations are equivalent to spherical latin bitrades, *J. Combin. Theory Ser. A* **15** (2008), 193–197.
- [5] N. J. Cavenagh and I. M. Wanless: Latin trades in groups defined on planar triangulations, *Journal of Algebraic Combinatorics* (in print).
- [6] C. J. Colbourn, J. H. Dinitz and I. M. Wanless: Latin squares, in *Handbook of combinatorial designs* (C. J. Colbourn and J. H. Dinitz editors). Second edition. Discrete Mathematics and its Applications (Boca Raton). Chapman & Hall/CRC, Boca Raton, FL, 2007.
- [7] A. Drápal: *Latin squares and partial groupoid*, CSc. thesis, Prague 1988 (in Czech).
- [8] A. Drápal and T. Kepka: Group modifications of some partial groupoids, *Annals of Discr. Math.* **18** (1983), 319–332.
- [9] A. Drápal and T. Kepka: Group distances of latin squares, *Comment. Math. Univ. Carolinae* **26** (1985), 275–289.
- [10] A. Drápal and T. Kepka: On a distance of groups and latin squares, *Comment. Math. Univ. Carolinae* **30** (1989), 621–626.
- [11] A. Drápal: On a planar construction of quasigroups, *Czechoslovak Math. J.* **41** (1991), 538–548.
- [12] A. Drápal: Hamming distances of groups and quasi-groups, *Discrete Math.* **235** (2001), 189–197.
- [13] A. Drápal: On geometrical structure and construction of latin trades, *Advances in geometry* (in print).
- [14] A. Drápal: On elementary moves that generate all spherical latin trades (submitted).
- [15] D. A. Holton, B. Manvel and B. D. McKay: Hamiltonian cycles in cubic 3-connected bipartite planar graphs, *J. Combin. Theory Ser. B* **38** (1985), 279–297.
- [16] A. D. Keedwell: Critical sets in squares and related matters: an update. *Utilitas Math.* **65** (2004), 97–131.
- [17] J. Lefevre, D. Donovan, N. Cavenagh and A. Drápal: Minimal and minimum size latin bitrades of each genus, *Comment. Math. Univ. Carolin.* **48** (2007), 189–203.

- [18] I. M. Wanless: A computer enumeration of small latin trades, *Australas. J. Combin.* **39** (2007), 247–258.